

Theory of Bessel Functions of HighRank - II: Hankel Transforms and Fundamental Bessel Kernels

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ABSTRACT. In this article we shall study the analytic theory and the representation theoretic interpretations of Hankel transforms and fundamental Bessel kernels of an arbitrary rank over an archimedean field.

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1. Introduction

In this article, we shall study *Hankel transforms*^I as well as their integral kernels, called *fundamental Bessel kernels*^{II}, over an archimedean field. These Hankel transforms are the archimedean constituent of the Voronoï summation formula over a number field.

1.1. Analytic theory. Let n be a positive integer. In the case $n \geq 3$ Hankel transforms of rank n over \mathbb{R} have been investigated in the work of Miller and Schmid [MS1, MS3, MS4] on the Voronoï summation formula for $\mathrm{GL}_n(\mathbb{Z})$. The notion of *automorphic distributions*^{III} is used for their proof of this formula, and is also used to derive the analytic continuation and the functional equation of the L -function of a cuspidal $\mathrm{GL}_n(\mathbb{Z})$ -automorphic representation of $\mathrm{GL}_n(\mathbb{R})$. As the foundation of automorphic distributions, the harmonic analysis over \mathbb{R} is studied in [MS3] from the viewpoint of the signed Mellin transforms. As explained in [MS2], the cases $n = 1, 2$ can also be incorporated into their framework. Furthermore, it is shown in the author's previous paper [Qi1] that all Hankel transforms over \mathbb{R} admit integral kernels, which can be partitioned into combinations of

^IThey are called Bessel transforms in some literatures, for instance, [IT]. However, this type of integral transforms should actually be attributed to Hermann Hankel. Moreover, we shall reserve the term *Bessel transforms* for the transforms shown in the Kuznetsov trace formula.

^{II}The adjective *fundamental* is added for the distinction from the Bessel functions for GL_n in the Kuznetsov formula, and will be dropped when no confusion occurs.

^{III}According to Stephen Miller, the origin of automorphic distributions can be traced back to the 19th century in the work of Siméon Poisson on Poisson's integral for harmonic functions on either the unit disk or the upper half-plane.

the so-called *fundamental Bessel functions*. These Bessel functions are studied from two approaches via their *formal integral representations* and *Bessel differential equations*.

In §2 - 8, we shall establish the analytic theory of Hankel transforms and their Bessel kernels over \mathbb{C} . The study of Hankel transforms for $\mathrm{GL}_n(\mathbb{C})$ from the perspective of [MS3] is complete to some extent. On the other hand, Bessel functions in [Qi1] play a fundamental role in our study of Bessel kernels over \mathbb{C} , for instance, in finding their asymptotic expansions. Although our main focus is on the theory over \mathbb{C} , the theory of Hankel transforms over \mathbb{R} extracted from [MS3] as well as some treatments of Bessel kernels over \mathbb{R} will also be included for the sake of comparison.

The sections §2 - 8 are outlined as follows.

In the preliminary section §2, some basic notions are introduced, such as gamma factors, Schwartz spaces, the Fourier transform and Mellin transforms. The three kinds of Mellin transforms \mathcal{M} , $\mathcal{M}_{\mathbb{R}}$ and $\mathcal{M}_{\mathbb{C}}$ are first defined over the Schwartz spaces over $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ respectively.

In §3, the definitions of the Mellin transforms \mathcal{M} , $\mathcal{M}_{\mathbb{R}}$ and $\mathcal{M}_{\mathbb{C}}$ are extended onto certain function spaces $\mathcal{S}_{\mathrm{sis}}(\mathbb{R}_+)$, $\mathcal{S}_{\mathrm{sis}}(\mathbb{R}^\times)$ and $\mathcal{S}_{\mathrm{sis}}(\mathbb{C}^\times)$ respectively. We shall precisely characterize their image spaces $\mathcal{M}_{\mathrm{sis}}$, $\mathcal{M}_{\mathrm{sis}}^{\mathbb{R}}$ and $\mathcal{M}_{\mathrm{sis}}^{\mathbb{C}}$ under their corresponding Mellin transforms. In spite of their similar constructions, the analysis of the Mellin transform $\mathcal{M}_{\mathbb{C}}$ is much more elaborate than that of $\mathcal{M}_{\mathbb{R}}$ or \mathcal{M} .

In §4, based on gamma factors and Mellin transforms, we shall construct Hankel transforms upon suitable subspaces of the $\mathcal{S}_{\mathrm{sis}}$ function spaces just introduced in §3 and study their Bessel kernels. It turns out that all these Bessel kernels can be formulated in terms of the Bessel functions in [Qi1].

In §5, we shall first introduce the Schmid-Miller transforms in companion with the Fourier transform and then use them to establish a Fourier type integral transform expression of a Hankel transform.

In §6, we shall introduce certain integrals, derived from the Fourier type integral transforms given in §5, that represents Bessel kernels. When the field is real, these integrals never absolutely converge and are closely connected to the formal integrals studied in [Qi1]. In the complex case, however, some range of index can be found where such integrals are absolutely convergent.

The last two sections §7 and §8 are devoted to Bessel kernels over \mathbb{C} . In §7, we shall prove two connection formulae that relate a Bessel kernel over \mathbb{C} to the two kinds of Bessel functions of *positive* sign. These kinds of Bessel functions arise in the study of Bessel equations in [Qi1, §7]. In §8, as a consequence of the second connection formula above, we shall derive the asymptotic expansion of a Bessel kernel over \mathbb{C} from [Qi1, Theorem 7.27].

1.2. Representation theory. The work of Miller and Schmid is extended by Ichino and Templier [IT] to any irreducible cuspidal automorphic representation of GL_n , $n \geq 2$, over an arbitrary number field \mathbb{K} . The Voronoï summation formula for GL_n follows from

the global theory of $\mathrm{GL}_n \times \mathrm{GL}_1$ -Rankin-Selberg L -functions. For an archimedean completion \mathbb{K}_v of \mathbb{K} , the defining identities of the Hankel transform associated with an infinite dimensional irreducible unitary generic representation of $\mathrm{GL}_n(\mathbb{K}_v)$ are reformulations of the corresponding local functional equations for $\mathrm{GL}_n \times \mathrm{GL}_1$ -Rankin-Selberg zeta integrals over \mathbb{K}_v .

In §9, we shall first recollect the definition of the Hankel transform associated with an infinite dimensional irreducible admissible generic representation of $\mathrm{GL}_n(\mathbb{F})$ for an archimedean field \mathbb{F} in [IT]. We stress that this definition actually works for any irreducible admissible representation of $\mathrm{GL}_n(\mathbb{F})$, including $n = 1$. We shall then give a detailed discussion on Hankel transforms of rank n over \mathbb{F} using the Langlands classification for $\mathrm{GL}_n(\mathbb{F})$.

In §10, according to the theory of local functional equations for $\mathrm{GL}_2 \times \mathrm{GL}_1$ -Rankin-Selberg zeta integrals over \mathbb{F} , we shall show that the action of the long Weyl element on the Kirillov model of an infinite dimensional irreducible admissible representation of $\mathrm{GL}_2(\mathbb{F})$ is essentially a Hankel transform over \mathbb{F} . It follows the consensus that for $\mathrm{GL}_2(\mathbb{F})$ the Bessel functions occurring in the Kuznetsov trace formula should coincide with those in the Voronoï summation formula. This will let us prove and generalize the Kuznetsov trace formula for $\mathrm{PSL}_2(\mathbb{Z}[i]) \backslash \mathrm{PSL}_2(\mathbb{C})$ in [BM]^{IV}, in the same way that [CPS] does for the Kuznetsov trace formula for $\mathrm{PSL}_2(\mathbb{Z}) \backslash \mathrm{PSL}_2(\mathbb{R})$ in [Kuz].

In [Qi2, §3.2], the author found a formula of Bessel functions for $\mathrm{GL}_3(\mathbb{F})$ in terms of fundamental Bessel kernels, formally derived from Rankin-Selberg $\mathrm{GL}_3 \times \mathrm{GL}_2$ local functional equations and the GL_2 Bessel-Plancherel formula. The Bessel functions for $\mathrm{GL}_3(\mathbb{F})$ are two-variable special functions arising in the Kuznetsov trace formula for $\mathrm{GL}_3(\mathbb{F})$. To be precise, the fundamental Bessel kernels occurring in the formula are for $\mathrm{GL}_3 \times \mathrm{GL}_2$ and GL_2 , where the involved representations of GL_2 are all tempered. This is the first instance of Bessel functions for groups of higher rank.

1.3. Distribution theory. Although such a theory can be formulated, we shall not touch in this article the theory of Hankel transforms over \mathbb{C} from the perspective of distributions as in [MS3]. It is very likely that this will lead to the theory of automorphic distributions on $\mathrm{GL}_n(\mathbb{C})$ with respect to congruence subgroups, as well as the Voronoï summation formula for cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{C})$. The Voronoï summation formula in this generality is already covered by [IT], but this approach would still be of its own interest.

^{IV}In [MW], Miatello and Wallach gave the spherical Kuznetsov trace formula for real semisimple groups of real rank one, which include both $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$. The first instance of Bessel functions for spherical principal series representations of $\mathrm{SL}_2(\mathbb{C})$ can be found in their work. More generally, in generalizing the Kuznetsov formula to the non-spherical case, the formula and the integral representation of the Bessel functions associated with principal series representations of $\mathrm{SL}_2(\mathbb{C})$ are discovered by Bruggeman, Motohashi and Lokvenec-Guleska [BM, LG]. Their approach is however entirely different from ours.

1.4. Applications. When $n = 2$, there are numerous applications in analytic number theory of the Voronoï summation formula and the Kuznetsov trace formula over \mathbb{Q} , which include subconvexity, non-vanishing of automorphic L -functions and estimates for shifted convolution sums. The Voronoï summation formula for $\mathrm{GL}_3(\mathbb{Z})$ is used to establish subconvexity in [Li] as well. In order to work over an arbitrary number field, one also needs to understand Hankel transforms and Bessel kernels at least for $\mathrm{GL}_2(\mathbb{C})$. We hope that the present work and its sequel will make these problems over a number field more approachable from the analytic perspective.

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2. Notations and preliminaries

2.1. General notations.

- Denote $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}_+ = \{1, 2, 3, \dots\}$.
- The group $\mathbb{Z}/2\mathbb{Z}$ is usually identified with the two-element set $\{0, 1\}$.
- Denote $\mathbb{R}_+ = (0, \infty)$, $\overline{\mathbb{R}}_+ = [0, \infty)$, $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.
- Denote by $\mathbb{U} \cong \mathbb{R}_+ \times \mathbb{R}$ the universal cover of $\mathbb{C} \setminus \{0\}$. Each element $\zeta \in \mathbb{U}$ is denoted by $\zeta = xe^{i\omega}$, with $(x, \omega) \in \mathbb{R}_+ \times \mathbb{R}$.
- For $m \in \mathbb{Z}$ define $\delta(m) \in \mathbb{Z}/2\mathbb{Z}$ by $\delta(m) = m \pmod{2}$.
- For $s \in \mathbb{C}$ and $\alpha \in \mathbb{N}$, let $[s]_\alpha = \prod_{k=0}^{\alpha-1} (s - \alpha)$ and $(s)_\alpha = \prod_{k=0}^{\alpha-1} (s + \alpha)$ if $\alpha \geq 1$, and let $[s]_0 = (s)_0 = 1$.
- For $s \in \mathbb{C}$ let $e(s) = e^{2\pi i s}$.
- For a finite closed interval $[a, b] \subset \mathbb{R}$ define the closed vertical strip $\mathbb{S}[a, b] = \{s \in \mathbb{C} : \Re s \in [a, b]\}$. The open vertical strip $\mathbb{S}(a, b)$ for a finite open interval (a, b) is similarly defined.
- For $\lambda \in \mathbb{C}$ and $r > 0$, define $\mathbb{B}_r(\lambda) = \{s \in \mathbb{C} : |s - \lambda| < r\}$ to be the disc of radius r centered at $s = \lambda$.
- For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ denote $|\lambda| = \sum_{l=1}^n \lambda_l$ (this notation works for subsets of \mathbb{C}^n , for instance, $(\mathbb{Z}/2\mathbb{Z})^n = \{0, 1\}^n$ and \mathbb{Z}^n).
- Define the hyperplane $\mathbb{L}^{n-1} = \{\lambda \in \mathbb{C}^n : |\lambda| = \sum_{l=1}^n \lambda_l = 0\}$.
- Denote by \mathbf{e}^n the n -tuple $(1, \dots, 1)$.
- For $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ define $\|\mathbf{m}\| = (|m_1|, \dots, |m_n|)$.

2.2. Gamma factors.

2.2.1. We define the gamma factor

$$(2.1) \quad G_\pm(s) = \Gamma(s) e\left(\pm \frac{s}{4}\right).$$

For $(\mathbf{s}, \boldsymbol{\lambda}) = (\varsigma_1, \dots, \varsigma_n, \lambda_1, \dots, \lambda_n) \in \{+, -\}^n \times \mathbb{C}^n$ let

$$(2.2) \quad G(s; \mathbf{s}, \boldsymbol{\lambda}) = \prod_{l=1}^n G_{\varsigma_l}(s - \lambda_l).$$

2.2.2. For $\delta \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, we define the gamma factor

$$(2.3) \quad G_\delta(s) = i^\delta \pi^{\frac{1}{2}-s} \frac{\Gamma\left(\frac{1}{2}(s + \delta)\right)}{\Gamma\left(\frac{1}{2}(1 - s + \delta)\right)} = \begin{cases} 2(2\pi)^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right), & \text{if } \delta = 0, \\ 2i(2\pi)^{-s} \Gamma(s) \sin\left(\frac{\pi s}{2}\right), & \text{if } \delta = 1. \end{cases}$$

Here, we have used the duplication formula and Euler's reflection formula of the Gamma function,

$$\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}, \quad \Gamma(s)\Gamma\left(s + \frac{1}{2}\right) = 2^{1-2s} \sqrt{\pi} \Gamma(2s).$$

Let $(\boldsymbol{\mu}, \boldsymbol{\delta}) = (\mu_1, \dots, \mu_n, \delta_1, \dots, \delta_n) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ and define

$$(2.4) \quad G_{(\boldsymbol{\mu}, \boldsymbol{\delta})}(s) = \prod_{l=1}^n G_{\delta_l}(s - \mu_l).$$

One observes the following simple functional relation

$$(2.5) \quad G_{(\boldsymbol{\mu}, \boldsymbol{\delta})}(1-s) G_{(-\boldsymbol{\mu}, \boldsymbol{\delta})}(s) = 1.$$

2.2.3. For $m \in \mathbb{Z}$, we define the gamma factor

$$(2.6) \quad G_m(s) = i^{|m|} (2\pi)^{1-2s} \frac{\Gamma\left(s + \frac{1}{2}|m|\right)}{\Gamma\left(1-s + \frac{1}{2}|m|\right)}.$$

Let $(\boldsymbol{\mu}, \mathbf{m}) = (\mu_1, \dots, \mu_n, m_1, \dots, m_n) \in \mathbb{C}^n \times \mathbb{Z}^n$ and define

$$(2.7) \quad G_{(\boldsymbol{\mu}, \mathbf{m})}(s) = \prod_{l=1}^n G_{m_l}(s - \mu_l).$$

We have the functional relation

$$(2.8) \quad G_{(\boldsymbol{\mu}, \mathbf{m})}(1-s) G_{(-\boldsymbol{\mu}, \mathbf{m})}(s) = 1.$$

2.2.4. *Relations between the three types of gamma factors.* We first observe that

$$G_\delta(s) = (2\pi)^{-s} (G_+(s) + (-)^\delta G_-(s)).$$

Hence

$$(2.9) \quad G_{(\boldsymbol{\mu}, \boldsymbol{\delta})}(s) = \sum_{\boldsymbol{\varsigma} \in \{+, -\}^n} \boldsymbol{\varsigma}^\delta (2\pi)^{|\boldsymbol{\mu}| - ns} G(s; \boldsymbol{\varsigma}, \boldsymbol{\mu}), \quad \boldsymbol{\varsigma}^\delta = \prod_{l=1}^n \varsigma_l^{\delta_l}, \quad |\boldsymbol{\mu}| = \sum_{l=1}^n \mu_l.$$

Euler's reflection formula and certain trigonometric identities yield

$$\begin{aligned}
 iG_m(s) &= i^{|m|+1} 2(2\pi)^{-2s} \Gamma\left(s + \frac{|m|}{2}\right) \Gamma\left(s - \frac{|m|}{2}\right) \sin\left(\pi\left(s - \frac{|m|}{2}\right)\right) \\
 (2.10) \quad &= G_{\delta(m)+1}\left(s - \frac{|m|}{2}\right) G_0\left(s + \frac{|m|}{2}\right) \\
 &= G_{\delta(m)}\left(s - \frac{|m|}{2}\right) G_1\left(s + \frac{|m|}{2}\right),
 \end{aligned}$$

with $\delta(m) = m \pmod{2}$. Therefore, $G_{(\mu, m)}(s)$ may be viewed as a certain $G_{(\eta, \delta)}(s)$ of doubled rank.

LEMMA 2.1. *Suppose that $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$ and $(\eta, \delta) \in \mathbb{C}^{2n} \times (\mathbb{Z}/2\mathbb{Z})^{2n}$ are subjected to one of the following two sets of relations*

$$(2.11) \quad \eta_{2l-1} = \mu_l + \frac{|m_l|}{2}, \quad \eta_{2l} = \mu_l - \frac{|m_l|}{2}, \quad \delta_{2l-1} = \delta(m) + 1, \quad \delta_{2l} = 0;$$

$$(2.12) \quad \eta_{2l-1} = \mu_l + \frac{|m_l|}{2}, \quad \eta_{2l} = \mu_l - \frac{|m_l|}{2}, \quad \delta_{2l-1} = \delta(m), \quad \delta_{2l} = 1.$$

Then $i^n G_{(\mu, m)}(s) = G_{(\eta, \delta)}(s)$.

2.2.5. *Stirling's formula.* Fix $s_0 \in \mathbb{C}$, and let $|\arg s| < \pi - \epsilon$, $0 < \epsilon < \pi$. We have the following asymptotic as $|s| \rightarrow \infty$

$$\log \Gamma(s_0 + s) \sim \left(s_0 + s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log(2\pi).$$

If one writes $s_0 = \varrho_0 + it_0$ and $s = \varrho + it$, $\varrho \geq 0$, then the right hand side is equal to

$$\begin{aligned}
 &\left(\varrho_0 + \varrho - \frac{1}{2}\right) \log \sqrt{t^2 + \varrho^2} - (t_0 + t) \arctan\left(\frac{t}{\varrho}\right) - \varrho + \frac{1}{2} \log(2\pi) \\
 &+ i(t_0 + t) \log \sqrt{t^2 + \varrho^2} - it + i\left(\varrho_0 + \varrho - \frac{1}{2}\right) \arctan\left(\frac{t}{\varrho}\right),
 \end{aligned}$$

and therefore

$$(2.13) \quad |\Gamma(s_0 + s)| \sim \sqrt{2\pi} (t^2 + \varrho^2)^{\frac{1}{2}(\varrho_0 + \varrho - \frac{1}{2})} e^{-(t_0 + t) \arctan(\frac{t}{\varrho}) - \varrho}.$$

LEMMA 2.2. *We have*

$$(2.14) \quad G(s; \mathfrak{s}, \lambda) \ll_{\lambda, a, b, r} (|\Im s| + 1)^{n(\Re s - \frac{1}{2}) - \Re |\lambda|},$$

for all $s \in \mathbb{S}[a, b] \setminus \bigcup_{l=1}^n \bigcup_{\kappa \in \mathbb{N}} \mathbb{B}_r(\lambda_l - \kappa)$, with small $r > 0$,

$$(2.15) \quad G_{(\mu, \delta)}(s) \ll_{\mu, a, b, r} (|\Im s| + 1)^{n(\Re s - \frac{1}{2}) - \Re |\mu|},$$

for all $s \in \mathbb{S}[a, b] \setminus \bigcup_{l=1}^n \bigcup_{\kappa \in \mathbb{N}} \mathbb{B}_r(\mu_l - \delta_l - 2\kappa)$, and

$$(2.16) \quad G_{(\mu, m)}(s) \ll_{\mu, a, b, r} \prod_{l=1}^n (|\Im s| + |m_l| + 1)^{2\Re s - 2\Re \mu_l - 1},$$

for all $2s \in \mathbb{S}[a, b] \setminus \bigcup_{l=1}^n \bigcup_{\kappa \in \mathbb{N}} \mathbb{B}_r(2\mu_l - |m_l| - 2\kappa)$.

In other words, if λ and μ are given, then $G(s; \mathfrak{S}, \lambda)$, $G_{(\mu, \delta)}(s)$ and $G_{(\mu, m)}(s)$ are all of moderate growth with respect to $\Im s$, uniformly on vertical strips (with bounded width), and moreover $G_{(\mu, m)}(s)$ is also of uniform moderate growth with respect to m .

2.3. Basic notions for \mathbb{R}_+ , \mathbb{R}^\times and \mathbb{C}^\times . Define $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. We observe the isomorphisms $\mathbb{R}^\times \cong \mathbb{R}_+ \times \{+, -\}$ ($\cong \mathbb{R}_+ \times \mathbb{Z}/2\mathbb{Z}$) and $\mathbb{C}^\times \cong \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$, the latter being realized via the *polar coordinates* $z = xe^{i\phi}$.

2.3.1. Let $||$ denote the ordinary absolute value on either \mathbb{R} or \mathbb{C} , and set $||_{\mathbb{R}} = ||$ for \mathbb{R} and $|| = ||_{\mathbb{C}} = |^2$ for \mathbb{C} . Let dx be the Lebesgue measure on \mathbb{R} , and let $d^\times x = |x|^{-1}dx$ be the standard choice of the multiplicative Haar measure on \mathbb{R}^\times . Similarly, let dz be twice the ordinary Lebesgue measure on \mathbb{C} , and choose the standard multiplicative Haar measure $d^\times z = ||z||^{-1}dz$ on \mathbb{C}^\times . Moreover, in the polar coordinates, one has $d^\times z = 2d^\times x d\phi$. For $x \in \mathbb{R}^\times$ the sign function $\text{sgn}(x)$ is equal to $x/|x|$, whereas for $z \in \mathbb{C}^\times$ we introduce the notation $[z] = z/|z|$.

Henceforth, we shall let \mathbb{F} be either \mathbb{R} or \mathbb{C} , and occasionally let x, y denote elements in \mathbb{F} even if $\mathbb{F} = \mathbb{C}$.

2.3.2. For $\delta \in \mathbb{Z}/2\mathbb{Z}$, we define the space $C_\delta^\infty(\mathbb{R}^\times)$ of all smooth functions $\varphi \in C^\infty(\mathbb{R}^\times)$ satisfying the parity condition

$$(2.17) \quad \varphi(-x) = (-)^\delta \varphi(x).$$

Observe that a function $\varphi \in C_\delta^\infty(\mathbb{R}^\times)$ is determined by its restriction on \mathbb{R}_+ , namely, $\varphi(x) = \text{sgn}(x)^\delta \varphi(|x|)$. Therefore,

$$(2.18) \quad C_\delta^\infty(\mathbb{R}^\times) = \text{sgn}(x)^\delta C^\infty(\mathbb{R}_+),$$

with the notation $\text{sgn}(x)^\delta C^\infty(\mathbb{R}_+) = \{\text{sgn}(x)^\delta \varphi(|x|) : \varphi \in C^\infty(\mathbb{R}_+)\}$.

For a smooth function $\varphi \in C^\infty(\mathbb{R}^\times)$, we define $\varphi_\delta \in C^\infty(\mathbb{R}_+)$ by

$$(2.19) \quad \varphi_\delta(x) = \frac{1}{2} (\varphi(x) + (-)^\delta \varphi(-x)), \quad x \in \mathbb{R}_+.$$

Clearly,

$$(2.20) \quad \varphi(x) = \varphi_0(|x|) + \text{sgn}(x) \varphi_1(|x|).$$

For $m \in \mathbb{Z}$, we define the space $C_m^\infty(\mathbb{C}^\times)$ of all smooth functions $\varphi \in C^\infty(\mathbb{C}^\times)$ satisfying

$$(2.21) \quad \varphi(xe^{i\phi} \cdot e^{i\phi'}) = e^{im\phi'} \varphi(xe^{i\phi}).$$

A function $\varphi \in C_m^\infty(\mathbb{C}^\times)$ is determined by its restriction on \mathbb{R}_+ , namely, $\varphi(z) = [z]^m \varphi(|z|)$, or, in the polar coordinates, $\varphi(xe^{i\phi}) = e^{im\phi} \varphi(x)$. Therefore,

$$(2.22) \quad C_m^\infty(\mathbb{R}^\times) = [z]^m C^\infty(\mathbb{R}_+),$$

with the notation $[z]^m C^\infty(\mathbb{R}_+) = \{[z]^m \varphi(|z|) = e^{im\phi} \varphi(x) : \varphi \in C^\infty(\mathbb{R}_+)\}$.

For a smooth function $\varphi \in C^\infty(\mathbb{R}^\times)$, we let $\varphi_m \in C^\infty(\mathbb{R}_+)$ denote the m -th Fourier coefficient of φ given by

$$(2.23) \quad \varphi_m(x) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(xe^{i\phi}) e^{-im\phi} d\phi.$$

One has the Fourier expansion of φ ,

$$(2.24) \quad \varphi(xe^{i\phi}) = \sum_{m \in \mathbb{Z}} \varphi_m(x) e^{im\phi}.$$

2.3.3. Subsequently, we shall encounter various subspaces of $C^\infty(\mathbb{F}^\times)$, with $\mathbb{F} = \mathbb{R}, \mathbb{C}$, for instance, $\mathcal{S}(\mathbb{F})$, $\mathcal{S}(\mathbb{F}^\times)$, $\mathcal{S}_{\text{sis}}(\mathbb{F}^\times)$, $\mathcal{S}_{\text{sis}}^{(\mu, \delta)}(\mathbb{R}^\times)$ and $\mathcal{S}_{\text{sis}}^{(\mu, m)}(\mathbb{C}^\times)$. Here, we list three central questions that will be the guidelines of our investigations of these function spaces.

For now, we let D be a subspace of $C^\infty(\mathbb{F}^\times)$. For $\mathbb{F} = \mathbb{R}$ (respectively $\mathbb{F} = \mathbb{C}$), we shall add a superscript or subscript δ (respectively m) to the notation of D , say D_δ (respectively D_m), to denote the space of $\varphi \in D$ satisfying (2.17) (respectively (2.21)). In view of (2.18) (respectively (2.22)), there is a subspace of $C^\infty(\mathbb{R}_+)$, say E_δ (respectively E_m), such that $D_\delta = \text{sgn}(x)^\delta E_\delta$ (respectively $D_m = [z]^m E_m$).

Firstly, we are interested in the question,

“How to characterize the space E_δ (respectively E_m)?”.

Moreover, the subspaces $D \subset C^\infty(\mathbb{F}^\times)$ that we shall consider always satisfy the following two hypotheses,

- $\varphi \in D$ implies $\varphi_\delta \in E_\delta$ for $\mathbb{F} = \mathbb{R}$ (respectively, $\varphi \in D$ implies $\varphi_m \in E_m$ for $\mathbb{F} = \mathbb{C}$), and
- D is closed under addition.

For $\mathbb{F} = \mathbb{R}$, under these two hypotheses, it follows from (2.20) that

$$D = D_0 \oplus D_1 \cong E_0 \times E_1.$$

For $\mathbb{F} = \mathbb{C}$, (2.24) implies that the map that sends φ to the sequence $\{\varphi_m\}_{m \in \mathbb{Z}}$ of its Fourier coefficients is injective. The second question arises,

“What is the image of D in $\prod_{m \in \mathbb{Z}} E_m$ under this map?”, or equivalently,

“What conditions should a sequence $\{\varphi_m\}_{m \in \mathbb{Z}} \in \prod_{m \in \mathbb{Z}} E_m$ satisfy in order for the Fourier series defined by (2.24) giving a function $\varphi \in D$?”.

Finally, after introducing the Mellin transform $\mathcal{M}_{\mathbb{F}}$, we shall focus on the question,

“What is the image of D under the Mellin transform $\mathcal{M}_{\mathbb{F}}$?”.

2.4. Schwartz spaces. We say that a function $\varphi \in C^\infty(\mathbb{R}_+)$ is *smooth at zero* if all of its derivatives admit asymptotics as below,

$$(2.25) \quad \varphi^{(\alpha)}(x) = \alpha! a_\alpha + O_\alpha(x) \text{ as } x \rightarrow 0, \text{ for any } \alpha \in \mathbb{N}, \text{ with } a_\alpha \in \mathbb{C}.$$

REMARK 2.3. Consequently, one has the asymptotic expansion $\varphi(x) \sim \sum_{k=0}^{\infty} a_k x^k$, which means that $\varphi(x) = \sum_{k=0}^A a_k x^k + O_A(x^{A+1})$ as $x \rightarrow 0$ for any $A \in \mathbb{N}$. It is not required that the series $\sum_{k=0}^{\infty} a_k x^k$ be convergent for any $x \in \mathbb{R}^\times$.

Actually, (2.25) is equivalent to the following

$$(2.26) \quad \varphi^{(\alpha)}(x) = \sum_{k=\alpha}^{\alpha+A} a_k[k]_\alpha x^{k-\alpha} + O_{\alpha,A}(x^{A+1}) \text{ as } x \rightarrow 0, \text{ for any } \alpha, A \in \mathbb{N}.$$

Another observation is that, for a given constant $1 > \varrho > 0$, (2.25) is equivalent to the following seemingly weaker statement,

$$(2.27) \quad \varphi^{(\alpha)}(x) = \alpha! a_\alpha + O_{\alpha,\varrho}(x^\varrho) \text{ as } x \rightarrow 0, \text{ for any } \alpha \in \mathbb{N}, \text{ with } a_\alpha \in \mathbb{C}.$$

Let $C^\infty(\overline{\mathbb{R}}_+)$ denote the subspace of $C^\infty(\mathbb{R}_+)$ consisting of smooth functions on \mathbb{R}_+ that are also smooth at zero.

Let $\mathcal{S}(\overline{\mathbb{R}}_+)$ denote the space of functions in $C^\infty(\overline{\mathbb{R}}_+)$ that rapidly decay at infinity along with all of their derivatives. Let $\mathcal{S}(\mathbb{F})$ denote the Schwartz space on \mathbb{F} , with $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

Let $\mathcal{S}(\mathbb{R}_+)$ denote the space of Schwartz functions on \mathbb{R}_+ , that is, smooth functions on \mathbb{R}_+ whose derivatives rapidly decay at *both* zero and infinity. Similarly, we denote by $\mathcal{S}(\mathbb{F}^\times)$ the space of Schwartz functions on \mathbb{F}^\times .

The following lemma provides criteria for characterizing functions in these Schwartz spaces, especially functions in $\mathcal{S}(\mathbb{C})$ or $\mathcal{S}(\mathbb{C}^\times)$ in the polar coordinates. Its proof is left as an easy excise in analysis for the reader.

LEMMA 2.4. *Let notations be as above.*

(1.1). *Let $\varphi \in C^\infty(\overline{\mathbb{R}}_+)$ satisfy the asymptotics (2.25). Then $\varphi \in \mathcal{S}(\overline{\mathbb{R}}_+)$ if and only if φ also satisfies*

$$(2.28) \quad x^{\alpha+\beta} \varphi^{(\alpha)}(x) \ll_{\alpha,\beta} 1 \text{ for all } \alpha, \beta \in \mathbb{N}.$$

(1.2). *A smooth function φ on \mathbb{R}_+ belongs to $\mathcal{S}(\mathbb{R}_+)$ if and only if φ satisfies (2.28) with $\beta \in \mathbb{N}$ replaced by $\beta \in \mathbb{Z}$.*

Let $\varphi \in \mathcal{S}(\overline{\mathbb{R}}_+)$ and a_α be as in (2.25). Then $\varphi \in \mathcal{S}(\mathbb{R}_+)$ if and only if $a_\alpha = 0$ for all $\alpha \in \mathbb{N}$.

(2.1). *A smooth function φ on \mathbb{R}^\times extends to a function in $\mathcal{S}(\mathbb{R})$ if and only if*

- φ satisfies (2.28) with $x^{\alpha+\beta}$ replaced by $|x|^{\alpha+\beta}$, and
- all the derivatives of φ admit asymptotics

$$(2.29) \quad \varphi^{(\alpha)}(x) = \alpha! a_\alpha + O_\alpha(|x|) \text{ as } x \rightarrow 0, \text{ for any } \alpha \in \mathbb{N}, \text{ with } a_\alpha \in \mathbb{C}.$$

(2.2). *Let φ be a smooth function on \mathbb{R}^\times . Then $\varphi \in \mathcal{S}(\mathbb{R}^\times)$ if and only if φ satisfies (2.28) with $x^{\alpha+\beta}$ replaced by $|x|^{\alpha+\beta}$ and $\beta \in \mathbb{N}$ by $\beta \in \mathbb{Z}$.*

Suppose $\varphi \in \mathcal{S}(\mathbb{R})$, then $\varphi \in \mathcal{S}(\mathbb{R}^\times)$ if and only if $\varphi^{(\alpha)}(0) = 0$ for all $\alpha \in \mathbb{N}$, or equivalently, $a_\alpha = 0$ for all $\alpha \in \mathbb{N}$, with a_α given in (2.29).

(3.1). Write $\partial_x = \partial/\partial x$ and $\partial_\phi = \partial/\partial \phi$. In the polar coordinates, a smooth function $\varphi(xe^{i\phi}) \in C^\infty(\mathbb{C}^\times)$ extends to a function in $\mathcal{S}(\mathbb{C})$ if and only if

- $\varphi(xe^{i\phi})$ satisfies

$$(2.30) \quad x^{\alpha+\beta} \partial_x^\alpha \partial_\phi^\beta \varphi(xe^{i\phi}) \ll_{\alpha,\beta,\gamma} 1 \text{ for all } \alpha, \beta, \gamma \in \mathbb{N},$$

- all the partial derivatives of φ admit asymptotics

$$(2.31) \quad x^\alpha \partial_x^\alpha \partial_\phi^\beta \varphi(xe^{i\phi}) = \sum_{|m| \leq \alpha+\beta} \sum_{\substack{|m| \leq \kappa \leq \alpha+\beta \\ \kappa \equiv m \pmod{2}}} a_{m,\kappa} [\kappa]_\alpha (im)^\beta x^\kappa e^{im\phi} + O_{\alpha,\beta}(x^{\alpha+\beta+1})$$

as $x \rightarrow 0$, for any $\alpha, \beta \in \mathbb{N}$, with $a_{m,\kappa} \in \mathbb{C}$ for $\kappa \geq |m|$ and $\kappa \equiv m \pmod{2}$.

Let $\varphi \in \mathcal{S}(\mathbb{C})$ and φ_m be the m -th Fourier coefficient of φ given by (2.23), then it follows from (2.30, 2.31) that

- φ_m satisfies

$$(2.32) \quad x^{\alpha+\beta} \varphi_m^{(\alpha)}(x) \ll_{\alpha,\beta,A} (|m| + 1)^{-A} \text{ for all } \alpha, \beta, A \in \mathbb{N},$$

- all the derivatives of φ_m admit asymptotics

$$(2.33) \quad \varphi_m^{(\alpha)}(x) = \sum_{\kappa=\alpha}^{\alpha+A} a_{m,\kappa} [\kappa]_\alpha x^{\kappa-\alpha} + O_{\alpha,A}((|m| + 1)^{-A} x^{A+1})$$

as $x \rightarrow 0$, for any given $\alpha, A \in \mathbb{N}$, with $a_{m,\kappa} \in \mathbb{C}$ satisfying $a_{m,\kappa} = 0$ if either $\kappa < |m|$ or $\kappa \not\equiv m \pmod{2}$.

Observe that (2.33) is equivalent to the following two conditions,

$$(2.34) \quad \varphi_m^{(\alpha)}(x) = \alpha! a_{m,\alpha} + O_\alpha(x) \text{ as } x \rightarrow 0, \text{ for any } \alpha \geq |m|, \text{ with } a_{m,\alpha} \in \mathbb{C} \text{ satisfying } a_{m,\alpha} = 0 \text{ if } \alpha \not\equiv m \pmod{2},$$

$$(2.35) \quad \text{for any given } \alpha, A \in \mathbb{N}, \varphi_m^{(\alpha)}(x) = O_{\alpha,A}((|m| + 1)^{-A} x^{A+1}) \text{ as } x \rightarrow 0, \text{ if } |m| > \alpha + A.$$

In particular, $\varphi_m \in \mathcal{S}(\overline{\mathbb{R}}_+)$.

Conversely, if a sequence $\{\varphi_m\}_{m \in \mathbb{Z}}$ of functions in $C^\infty(\mathbb{R}_+)$ satisfies (2.32), (2.34) and (2.35), then the Fourier series defined by $(\varphi_m)_{m \in \mathbb{Z}}$, that is, the right hand side of (2.24), is a Schwartz function on \mathbb{C} .

(3.2). In the polar coordinates, a smooth function $\varphi(xe^{i\phi}) \in C^\infty(\mathbb{C}^\times)$ is a Schwartz function on \mathbb{C}^\times if and only if φ satisfies (2.30) with $\beta \in \mathbb{N}$ by $\beta \in \mathbb{Z}$.

Let $\varphi \in \mathcal{S}(\mathbb{C}^\times)$ and φ_m be the m -th Fourier coefficient of φ , then it is necessary that φ_m satisfies (2.32) with $\beta \in \mathbb{N}$ replaced by $\beta \in \mathbb{Z}$. In particular, $\varphi_m \in \mathcal{S}(\mathbb{R}_+)$.

Conversely, if a sequence $\{\varphi_m\}_{m \in \mathbb{Z}}$ of functions in $C^\infty(\mathbb{R}_+)$ satisfies the condition (2.32) with $\beta \in \mathbb{N}$ replaced by $\beta \in \mathbb{Z}$, then the Fourier series defined by $\{\varphi_m\}_{m \in \mathbb{Z}}$ gives rise to a Schwartz function on \mathbb{C}^\times .

Let $\varphi \in \mathcal{S}(\mathbb{C})$ and $a_{m,\kappa}$ be given in (2.31), (2.33) or (2.34). $\varphi \in \mathcal{S}(\mathbb{C}^\times)$ if and only if $a_{m,\kappa} = 0$ for all $m \in \mathbb{Z}, \kappa \in \mathbb{N}$.

2.4.1. *Some subspaces of $\mathcal{S}(\overline{\mathbb{R}}_+)$.* In the following, we introduce several subspaces of $\mathcal{S}(\overline{\mathbb{R}}_+)$ that are closely related to $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}(\mathbb{C})$.

We first define for $\delta \in \mathbb{Z}/2\mathbb{Z}$ the subspace $C_\delta^\infty(\overline{\mathbb{R}}_+) \subset C^\infty(\overline{\mathbb{R}}_+)$ of functions with an asymptotic expansion of the form $\sum_{\kappa=0}^\infty a_\kappa x^{\delta+2\kappa}$ at zero.

REMARK 2.5. *A question arises, “whether $C^\infty(\overline{\mathbb{R}}_+) = C_0^\infty(\overline{\mathbb{R}}_+) + C_1^\infty(\overline{\mathbb{R}}_+)$?”.*

The answer is affirmative.

To see this, we define the space $C_\delta^\infty(\mathbb{R})$ of smooth functions φ on \mathbb{R} satisfying (2.17). One has $\text{sgn}(x)^\delta \varphi(|x|) \in C_\delta^\infty(\mathbb{R})$ if $\varphi \in C_\delta^\infty(\overline{\mathbb{R}}_+)$, and conversely, $\varphi|_{\mathbb{R}_+} \in C_\delta^\infty(\overline{\mathbb{R}}_+)$ if $\varphi \in C_\delta^\infty(\mathbb{R})$. Thus, with the simple observation $C^\infty(\mathbb{R}) = C_0^\infty(\mathbb{R}) \oplus C_1^\infty(\mathbb{R})$, one sees that $C_0^\infty(\overline{\mathbb{R}}_+) + C_1^\infty(\overline{\mathbb{R}}_+)$ is the subspace of $C^\infty(\overline{\mathbb{R}}_+)$ consisting of functions on \mathbb{R}_+ that admit a smooth extension onto \mathbb{R} .

On the other hand, the Borel theorem ([Nar, 1.5.4]), which is a special case of the Whitney extension theorem ([Nar, 1.5.5, 1.5.6]), states that for any sequence $\{a_\alpha\}_{\alpha \in \mathbb{N}}$ of constants there exists a smooth function $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi^{(\alpha)}(0) = \alpha! a_\alpha$. Clearly, this theorem of Borel implies our assertion above.

In §3.1.3, we shall give an alternative proof of this using the Mellin transform. See Remark 3.5.

We define $\mathcal{S}_\delta(\overline{\mathbb{R}}_+) = \mathcal{S}(\overline{\mathbb{R}}_+) \cap C_\delta^\infty(\overline{\mathbb{R}}_+)$. The following identity is obvious

$$\mathcal{S}_\delta(\overline{\mathbb{R}}_+) = x^\delta \mathcal{S}_0(\overline{\mathbb{R}}_+).$$

In view of Lemma 2.4 (1.2), one has $\mathcal{S}_0(\overline{\mathbb{R}}_+) \cap \mathcal{S}_1(\overline{\mathbb{R}}_+) = \mathcal{S}(\mathbb{R}_+)$.

If we let $\mathcal{S}_\delta(\mathbb{R})$ be the space of functions $\varphi \in \mathcal{S}(\mathbb{R})$ satisfying (2.17), then

$$\mathcal{S}_\delta(\mathbb{R}) = \text{sgn}(x)^\delta \mathcal{S}_\delta(\overline{\mathbb{R}}_+) = \left\{ \text{sgn}(x)^\delta \varphi(|x|) : \varphi \in \mathcal{S}_\delta(\overline{\mathbb{R}}_+) \right\}.$$

Clearly, $\mathcal{S}(\mathbb{R}) = \mathcal{S}_0(\mathbb{R}) \oplus \mathcal{S}_1(\mathbb{R})$.

We define the subspace $\mathcal{S}_m(\overline{\mathbb{R}}_+) \subset \mathcal{S}_{\delta(m)}(\overline{\mathbb{R}}_+)$, with $\delta(m) = m \pmod{2}$, of functions with an asymptotic expansion of the form $\sum_{\kappa=0}^\infty a_\kappa x^{|m|+2\kappa}$ at zero. We have

$$\mathcal{S}_m(\overline{\mathbb{R}}_+) = x^{|m|} \mathcal{S}_0(\overline{\mathbb{R}}_+).$$

If we define $\mathcal{S}_m(\mathbb{C})$ to be the space of $\varphi \in \mathcal{S}(\mathbb{C})$ satisfying (2.21), then

$$\mathcal{S}_m(\mathbb{C}) = [z]^m \mathcal{S}_m(\overline{\mathbb{R}}_+) = \left\{ [z]^m \varphi(|z|) = e^{im\phi} \varphi(x) : \varphi \in \mathcal{S}_m(\overline{\mathbb{R}}_+) \right\}.$$

The last two paragraphs in Lemma 2.4 (3.1) can be recapitulated as below

$$\mathcal{S}(\mathbb{C}) \xrightarrow{\cong} \left\{ \{\varphi_m\} \in \prod_{m \in \mathbb{Z}} \mathcal{S}_m(\overline{\mathbb{R}}_+) : \varphi_m \text{ satisfies (2.32, 2.34, 2.35)} \right\} \twoheadrightarrow \mathcal{S}_m(\overline{\mathbb{R}}_+),$$

where the first map sends $\varphi \in \mathcal{S}(\mathbb{C})$ to the sequence $\{\varphi_m\}_{m \in \mathbb{Z}}$ of its Fourier coefficients, and the second is the m -th projection. According to Lemma 2.4 (3.1), the first map is an isomorphism, and the second projection is surjective.

2.4.2. $\mathcal{S}_\delta(\mathbb{R}^\times)$ and $\mathcal{S}_m(\mathbb{C}^\times)$. Let $\delta \in \mathbb{Z}/2\mathbb{Z}$ and $m \in \mathbb{Z}$. We define $\mathcal{S}_\delta(\mathbb{R}^\times) = \mathcal{S}(\mathbb{R}^\times) \cap \mathcal{S}_\delta(\mathbb{R})$ and $\mathcal{S}_m(\mathbb{C}^\times) = \mathcal{S}(\mathbb{C}^\times) \cap \mathcal{S}_m(\mathbb{C})$. Clearly, $\mathcal{S}_\delta(\mathbb{R}^\times) = \text{sgn}(x)^\delta \mathcal{S}(\mathbb{R}_+)$ and $\mathcal{S}_m(\mathbb{C}^\times) = [z]^m \mathcal{S}(\mathbb{R}_+)$.

2.5. The Fourier transform. According to the local theory in Tate's thesis for an archimedean local field \mathbb{F} , the Fourier transform $\hat{\varphi} = \mathcal{F}\varphi$ of a Schwartz function $\varphi \in \mathcal{S}(\mathbb{F})$ is defined by

$$(2.36) \quad \hat{\varphi}(y) = \int_{\mathbb{F}} \varphi(x) e(-\Lambda(xy)) dx,$$

with

$$(2.37) \quad \Lambda(x) = \begin{cases} x, & \text{if } \mathbb{F} = \mathbb{R}; \\ \text{Tr}(x) = (x + \bar{x}), & \text{if } \mathbb{F} = \mathbb{C}. \end{cases}$$

The Schwartz space $\mathcal{S}(\mathbb{F})$ is invariant under the Fourier transform. Moreover, with our choice of measure in §2.3, the following inversion formula holds

$$(2.38) \quad \hat{\hat{\varphi}}(x) = \varphi(-x), \quad x \in \mathbb{F}.$$

2.6. The Mellin transforms \mathcal{M} , \mathcal{M}_δ and \mathcal{M}_m . Corresponding to \mathbb{R}_+ , \mathbb{R}^\times and \mathbb{C}^\times , there are three kinds of Mellin transforms \mathcal{M} , \mathcal{M}_δ and \mathcal{M}_m .

DEFINITION 2.6 (Mellin transforms).

(1). The Mellin transform $\mathcal{M}\varphi$ of a Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}_+)$ is given by

$$(2.39) \quad \mathcal{M}\varphi(s) = \int_{\mathbb{R}_+} \varphi(x) x^s d^\times x.$$

(2). For $\delta \in \mathbb{Z}/2\mathbb{Z}$, the (signed) Mellin transform $\mathcal{M}_\delta\varphi$ with order δ of a Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}^\times)$ is defined by

$$(2.40) \quad \mathcal{M}_\delta\varphi(s) = \int_{\mathbb{R}^\times} \varphi(x) \text{sgn}(x)^\delta |x|^s d^\times x.$$

Moreover, define $\mathcal{M}_{\mathbb{R}} = (\mathcal{M}_0, \mathcal{M}_1)$.

(3). For $m \in \mathbb{Z}$, the Mellin transform $\mathcal{M}_m\varphi$ with order m of a Schwartz function $\varphi \in \mathcal{S}(\mathbb{C}^\times)$ is defined by

$$(2.41) \quad \mathcal{M}_m\varphi(s) = \int_{\mathbb{C}^\times} \varphi(z) [z]^m \|z\|^{\frac{1}{2}s} d^\times z = 2 \int_0^\infty \int_0^{2\pi} \varphi(xe^{i\phi}) e^{im\phi} d\phi \cdot x^s d^\times x.$$

Moreover, define $\mathcal{M}_{\mathbb{C}} = \prod_{m \in \mathbb{Z}} \mathcal{M}_{-m}$.

OBSERVATION 2.1. For $\varphi \in \mathcal{S}(\mathbb{R}^\times)$, we have

$$(2.42) \quad \mathcal{M}_\delta\varphi(s) = 2\mathcal{M}\varphi_\delta(s), \quad \delta \in \mathbb{Z}/2\mathbb{Z}.$$

Similarly, for $\varphi \in \mathcal{S}(\mathbb{C}^\times)$, we have

$$(2.43) \quad \mathcal{M}_{-m}\varphi(s) = 4\pi\mathcal{M}\varphi_m(s), \quad m \in \mathbb{Z}.$$

The relations (2.42) and (2.43) reflect the identities $\mathbb{R}^\times \cong \mathbb{R}_+ \times \{+, -\}$ and $\mathbb{C}^\times \cong \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$ respectively.

LEMMA 2.7 (Mellin inversions). *Let $\sigma \in \mathbb{R}$. Denote by (σ) the vertical line from $\sigma - i\infty$ to $\sigma + i\infty$.*

(1). *For $\varphi \in \mathcal{S}(\mathbb{R}_+)$, we have*

$$(2.44) \quad \varphi(x) = \frac{1}{2\pi i} \int_{(\sigma)} \mathcal{M}\varphi(s) x^{-s} ds.$$

(2). *For $\varphi \in \mathcal{S}(\mathbb{R}^\times)$, we have*

$$(2.45) \quad \varphi(x) = \frac{1}{4\pi i} \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \text{sgn}(x)^\delta \int_{(\sigma)} \mathcal{M}_\delta \varphi(s) |x|^{-s} ds.$$

(3). *For $\varphi \in \mathcal{S}(\mathbb{C}^\times)$, we have*

$$(2.46) \quad \varphi(z) = \frac{1}{8\pi^2 i} \sum_{m \in \mathbb{Z}} [z]^{-m} \int_{(\sigma)} \mathcal{M}_m \varphi(s) \|z\|^{-\frac{1}{2}s} ds,$$

or, in the polar coordinates,

$$(2.47) \quad \varphi(xe^{i\phi}) = \frac{1}{8\pi^2 i} \sum_{m \in \mathbb{Z}} e^{-im\phi} \int_{(\sigma)} \mathcal{M}_m \varphi(s) x^{-s} ds.$$

DEFINITION 2.8.

(1). *Let \mathcal{H}_{rd} denote the space of all entire functions $H(s)$ on the complex plane that rapidly decay along vertical lines, uniformly on vertical strips.*

(2). *Define $\mathcal{H}_{\text{rd}}^{\mathbb{R}} = \mathcal{H}_{\text{rd}} \times \mathcal{H}_{\text{rd}}$.*

(3). *Let $\mathcal{H}_{\text{rd}}^{\mathbb{C}}$ be the subset of $\prod_{\mathbb{Z}} \mathcal{H}_{\text{rd}}$ consisting of sequences $\{H_m\}_{m \in \mathbb{Z}}$ of entire functions in \mathcal{H}_{rd} satisfying,*

(2.48) *for any given $\alpha, A \in \mathbb{N}$ and vertical strip $\mathbb{S}[a, b]$,*

$$H_m(s) \ll_{\alpha, A, a, b} (|m| + 1)^{-A} (|\Im s| + 1)^{-\alpha} \text{ for all } s \in \mathbb{S}[a, b].$$

COROLLARY 2.9.

(1). *The Mellin transform \mathcal{M} and its inversion establish an isomorphism between $\mathcal{S}(\mathbb{R}_+)$ and \mathcal{H}_{rd} .*

(2). *For each $\delta \in \mathbb{Z}/2\mathbb{Z}$, \mathcal{M}_δ establishes an isomorphism between $\mathcal{S}_\delta(\mathbb{R}^\times)$ and \mathcal{H}_{rd} . Hence, $\mathcal{M}_{\mathbb{R}}$ establishes an isomorphism between $\mathcal{S}(\mathbb{R}^\times)$ and $\mathcal{H}_{\text{rd}}^{\mathbb{R}}$.*

(3). *For each $m \in \mathbb{Z}$, \mathcal{M}_{-m} establishes an isomorphism between $\mathcal{S}_m(\mathbb{C}^\times)$ and \mathcal{H}_{rd} . Moreover, $\mathcal{M}_{\mathbb{C}}$ establishes an isomorphism between $\mathcal{S}(\mathbb{C}^\times)$ and $\mathcal{H}_{\text{rd}}^{\mathbb{C}}$.*

PROOF. (1) is a well-known consequence of Lemma 2.7 (1), whereas (2) directly follows from (1) and Lemma 2.7 (2). As for (3), in addition to (1) and Lemma 2.7 (3), Lemma 2.4 (3.2) is also required. Q.E.D.

3. The function spaces $\mathcal{S}_{\text{sis}}(\mathbb{R}_+)$, $\mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$ and $\mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$

The goal of this section is to extend the definitions of the Mellin transforms \mathcal{M} , $\mathcal{M}_{\mathbb{F}}$ and generalize the settings in §2.6 to the function spaces $\mathcal{S}_{\text{sis}}(\mathbb{R}_+)$, \mathcal{M}_{sis} , $\mathcal{S}_{\text{sis}}(\mathbb{F}^\times)$ and $\mathcal{M}_{\text{sis}}^{\mathbb{F}}$. These spaces are much more sophisticated than $\mathcal{S}(\mathbb{R}_+)$, \mathcal{H}_{rd} , $\mathcal{S}(\mathbb{F}^\times)$ and $\mathcal{H}_{\text{rd}}^{\mathbb{F}}$ but most suitable for investigating Hankel transforms over \mathbb{R}_+ and \mathbb{F}^\times .

We shall first construct the function spaces $\mathcal{S}_{\text{sis}}(\mathbb{R}_+)$, \mathcal{M}_{sis} and establish an isomorphism between them using the Mellin transform \mathcal{M} . Based on these, we shall then turn to the spaces $\mathcal{S}_{\text{sis}}(\mathbb{F}^\times)$, $\mathcal{M}_{\text{sis}}^{\mathbb{F}}$ and the Mellin transform $\mathcal{M}_{\mathbb{F}}$. The case $\mathbb{F} = \mathbb{R}$ has been worked out in [MS3, §6]. Since $\mathbb{R}^\times \cong \mathbb{R}_+ \times \{+, -\}$ is simply two copies of \mathbb{R}_+ , the properties of $\mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$ and $\mathcal{M}_{\text{sis}}^{\mathbb{R}}$ are in substance the same as those of $\mathcal{S}_{\text{sis}}(\mathbb{R}_+)$ and \mathcal{M}_{sis} . In the case $\mathbb{F} = \mathbb{C}$, $\mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$ and $\mathcal{M}_{\text{sis}}^{\mathbb{C}}$ can be constructed in a parallel way. What underlies is the simple observation $\mathbb{C}^\times \cong \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$. The study, however, is much more elaborate, since the analysis on the circle $\mathbb{R}/2\pi\mathbb{Z}$ is also taken into account.

3.1. The spaces $\mathcal{S}_{\text{sis}}(\mathbb{R}_+)$ and \mathcal{M}_{sis} .^V

3.1.1. *The spaces $x^{-\lambda}(\log x)^j \mathcal{S}(\overline{\mathbb{R}}_+)$ and $\mathcal{M}_{\text{sis}}^{\lambda,j}$.* Let $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$.

We define

$$x^{-\lambda}(\log x)^j \mathcal{S}(\overline{\mathbb{R}}_+) = \left\{ x^{-\lambda}(\log x)^j \varphi(x) : \varphi \in \mathcal{S}(\overline{\mathbb{R}}_+) \right\}.$$

We say that a meromorphic function $H(s)$ has a pole of *pure order* $j+1$ at $s = \lambda$ if the principal part of $H(s)$ at $s = \lambda$ is $a(s - \lambda)^{-j-1}$ for some constant $a \in \mathbb{C}$. Of course, $H(s)$ does not have a genuine pole at $s = \lambda$ if $a = 0$. We define the space $\mathcal{M}_{\text{sis}}^{\lambda,j}$ of all meromorphic functions $H(s)$ on the complex plane such that

- the only possible singularities of H are poles of pure order $j+1$ at the points in $\lambda - \mathbb{N} = \{\lambda - \kappa : \kappa \in \mathbb{N}\}$, and
 - $H(s)$ decays rapidly along vertical lines, uniformly on vertical strips, that is,
- (3.1) for any given $\alpha \in \mathbb{N}$, vertical strip $\mathbb{S}[a, b]$ and $r > 0$,

$$H(s) \ll_{\lambda,j,\alpha,a,b,r} (|\Im s| + 1)^{-\alpha} \text{ for all } s \in \mathbb{S}[a, b] \setminus \bigcup_{\kappa \in \mathbb{N}} \mathbb{B}_r(\lambda - \kappa).$$

The constructions of the Mellin transform \mathcal{M} and its inversion (2.39, 2.44) identically extend from $\mathcal{S}(\mathbb{R}_+)$ onto $\mathcal{S}_{\text{sis}}^{\lambda,j}(\mathbb{R}_+)$, except that the conditions $\Re s > \Re \lambda$ and $\sigma > \Re \lambda$ are required to guarantee convergence.

LEMMA 3.1. *Let $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$. The Mellin transform \mathcal{M} and its inversion establish an isomorphism of between $x^{-\lambda}(\log x)^j \mathcal{S}(\overline{\mathbb{R}}_+)$ and $\mathcal{M}_{\text{sis}}^{\lambda,j}$.*

This lemma is essentially [MS3, Lemma 6.13, Corollary 6.17]. Nevertheless, we shall include its proof as the reference for the constructions of $\mathcal{N}_{\text{sis}}^{\mathbb{C},\lambda,j}$ and $\mathcal{M}_{\text{sis}}^{\mathbb{C}}$ in §3.3.2 as well as the proof of Lemma 3.8.

PROOF. Let $v(x) = x^{-\lambda}(\log x)^j \varphi(x)$ for some $\varphi \in \mathcal{S}(\overline{\mathbb{R}}_+)$. Suppose that the derivatives of φ satisfy (2.26) and (2.28), that is, asymptotic expansions at zero and the Schwartz condition at infinity.

CLAIM 1. Let

$$H(s) = \mathcal{M}v(s) = \int_0^\infty v(s) x^{s-1} dx, \quad \Re s > \Re \lambda.$$

^VAccording to [MS3, Definition 6.4], a function in $\mathcal{S}_{\text{sis}}(\mathbb{R}_+)$ is said to have a *simple singularity* at zero. Thus, the subscript “sis” stands for “simple singularity”.

Then H admits a meromorphic continuation onto the whole complex plane. The only singularities of H are poles of pure order $j+1$ at the points in $\lambda - \mathbb{N}$. More precisely, $H(s)$ has a pole at $s = \lambda - \kappa$ of principal part $(-)^j j! a_\kappa (s - \lambda + \kappa)^{-j-1}$. Moreover, H decays rapidly along vertical lines, uniformly on vertical strips. To be concrete, we have

(3.2) for any given $\alpha, A \in \mathbb{N}, b \geq a > \Re \lambda - \alpha - A - 1$ and $r > 0$,

$$H(s) \ll_{\lambda, j, \alpha, A, a, b, r} (|\Im s| + 1)^{-\alpha} \text{ for all } s \in \mathbb{S}[a, b] \setminus \bigcup_{\kappa=0}^{\alpha+A} \mathbb{B}_r(\lambda - \kappa).$$

We remark that (3.1) and (3.2) are equivalent.

PROOF OF CLAIM 1. In view of $\mathcal{M}(x^{-\lambda}(\log x)^j \varphi(x))(s) = \mathcal{M}((\log x)^j \varphi(x))(s - \lambda)$, one may assume $\lambda = 0$. As such, $v(x) = (\log x)^j \varphi(x)$.

Let $A \in \mathbb{N}$. We have for $\Re s > 0$

$$(3.3) \quad \begin{aligned} \mathcal{M}v(s) &= \int_0^1 (\log x)^j \left(\varphi(x) - \sum_{\kappa=0}^A a_\kappa x^\kappa \right) x^{s-1} dx + \sum_{\kappa=0}^A \frac{(-)^j j! a_\kappa}{(s + \kappa)^{j+1}} \\ &\quad + \int_1^\infty (\log x)^j \varphi(x) x^{s-1} dx. \end{aligned}$$

Here, we have used

$$\int_0^1 (\log x)^j x^{s-1} dx = \frac{(-)^j j!}{s^{j+1}}, \quad \Re s > 0.$$

In view of $\varphi(x) - \sum_{\kappa=0}^A a_\kappa x^\kappa = O_A(x^{A+1})$, the first integral in (3.3) converges in the half-plane $\{s : \Re s > -A - 1\}$. The last integral converges for all s on the whole complex plane due to the rapid decay of φ . Thus $H(s) = \mathcal{M}v(s)$ admits a meromorphic extension onto $\{s : \Re s > -A - 1\}$ and, since A was arbitrary, onto the whole complex plane, with poles of pure order $j+1$ at the points in $-\mathbb{N}$.

For any given $\alpha \in \mathbb{N}$, repeating partial integration α times to the defining integral of $\mathcal{M}v$ yields

$$(-)^\alpha(s)_\alpha \mathcal{M}v(s) = \mathcal{M}v^{(\alpha)}(s + \alpha).$$

If one expands $v^{(\alpha)}(x) = (d/dx)^\alpha ((\log x)^j \varphi(x))$ using the chain rule, writes each resulting terms of $\mathcal{M}v^{(\alpha)}(s + \alpha)$ after the expansion in the same fashion as (3.3), and applies (2.26) and (2.28) to estimate the first and the last integral respectively, then

$$\mathcal{M}v(s) \ll_{\alpha, A, a, b} \frac{1}{|(s)_\alpha|} \left(1 + \sum_{\kappa=0}^{\alpha+A} \left(\frac{1}{|s + \kappa|} + \dots + \frac{1}{|s + \kappa|^{j+1}} \right) \right),$$

for all $s \in \mathbb{S}[a, b]$, with $b \geq a > -\alpha - A - 1$. In particular, (3.2) is proven.

Let $H \in \mathcal{M}_{\text{sis}}^{\lambda, j}$. Suppose that the principal part of $H(s)$ at $s = \lambda - \kappa$ is equal to $(-)^j j! a_\kappa (s + \lambda + \kappa)^{-j-1}$ and that $H(s)$ satisfies the condition (3.2).

CLAIM 2. If we denote by $v(x)$ the following integral

$$(3.4) \quad v(x) = \frac{1}{2\pi i} \int_{(\sigma)} H(s) x^{-s} ds, \quad \sigma > \Re \lambda,$$

then all the derivatives of $\varphi(x) = x^\lambda(\log x)^{-j}v(x)$ satisfies the asymptotics in (2.27) at zero and rapidly decay at infinity.

PROOF OF CLAIM 2. Again, let us assume $\lambda = 0$.

Let $1 > \varrho > 0$. We left shift the contour of integration from (σ) to $(-\varrho)$. When moving across $s = 0$, we obtain $a_0(\log x)^j$ in view of Cauchy's differentiation formula^{VI}. It follows that

$$v(x) = a_0(\log x)^j + \frac{1}{2\pi i} \int_{(-\varrho)} H(s)x^{-s}ds.$$

Using (3.2) with r small, say $r < \varrho$, to estimate the above integral, one arrives at

$$v(x) = a_0(\log x)^j + O(x^\varrho) = (\log x)^j (a_0 + O(x^\varrho)), \text{ as } x \rightarrow 0.$$

Thus, $\varphi(x) = (\log x)^{-j}v(x)$ satisfies the asymptotic (2.27) with $\alpha = 0$. For the general case $\alpha \in \mathbb{N}$, we have

$$(3.5) \quad v^{(\alpha)}(x) = (-)^\alpha \frac{1}{2\pi i} \int_{(\sigma)} (s)_\alpha H(s)x^{-s-\alpha}ds.$$

Shifting the contour from (σ) to $(-\alpha - \varrho)$ and following the same line of arguments above, combined with some straightforward algebraic manipulations, one may show (2.27) by an induction.

We are left to show the Schwartz condition for $\varphi(x) = (\log x)^{-j}v(x)$, or equivalently, for $v(x)$. Indeed, the bound (2.28) for $v^{(\alpha)}(x)$ follows from right shifting the contour of the integral in (3.5) to the vertical line (β) and applying the estimates in (3.2). Q.E.D.

3.1.2. *The spaces $\mathcal{S}_{\text{sis}}(\mathbb{R}_+)$ and \mathcal{M}_{sis} .* Let $\lambda, \lambda' \in \mathbb{C}$. We write $\lambda \leq_1 \lambda'$ if $\lambda' - \lambda \in \mathbb{N}$ and $\lambda \sim_1 \lambda'$ if $\lambda' - \lambda \in \mathbb{Z}$. Observe that “ \leq_1 ” and “ \sim_1 ” define an order relation and an equivalence relation on \mathbb{C} respectively.

Define

$$\mathcal{S}_{\text{sis}}(\mathbb{R}_+) = \sum_{\lambda \in \mathbb{C}} \sum_{j \in \mathbb{N}} x^{-\lambda} (\log x)^j \mathcal{S}(\overline{\mathbb{R}}_+),$$

where the sum $\sum_{\lambda \in \mathbb{C}} \sum_{j \in \mathbb{N}}$ is in the *algebraic* sense. It is clear that $\lambda \leq_1 \lambda'$ if and only if $x^{-\lambda} (\log x)^j \mathcal{S}(\overline{\mathbb{R}}_+) \subseteq x^{-\lambda'} (\log x)^j \mathcal{S}(\overline{\mathbb{R}}_+)$. One also observes that $x^{-\lambda} (\log x)^j \mathcal{S}(\overline{\mathbb{R}}_+) \cap x^{-\lambda'} (\log x)^{j'} \mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}(\mathbb{R}_+)$ if either $j \neq j'$ or $\lambda \not\sim_1 \lambda'$. Therefore,

$$(3.6) \quad \mathcal{S}_{\text{sis}}(\mathbb{R}_+)/\mathcal{S}(\mathbb{R}_+) = \bigoplus_{\omega \in \mathbb{C}/\sim_1} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} (x^{-\lambda} (\log x)^j \mathcal{S}(\overline{\mathbb{R}}_+))/\mathcal{S}(\mathbb{R}_+).$$

Here the direct limit $\varinjlim_{\lambda \in \omega}$ is taken on the totally ordered set (ω, \leq_1) and may be simply viewed as the union $\bigcup_{\lambda \in \omega}$. More precisely, each function $v \in \mathcal{S}_{\text{sis}}(\mathbb{R}_+)$ can be expressed

^{VI}Recall Cauchy's differentiation formula,

$$f^{(j)}(\zeta) = \frac{j!}{2\pi i} \oint_{\partial \mathbb{B}_r(\zeta)} \frac{f(s)}{(s - \zeta)^{j+1}} ds,$$

where f is a holomorphic function on a neighborhood of the closed disc $\overline{\mathbb{B}}_r(\zeta)$ centered at ζ , and the integral is taken counter-clockwise on the circle $\partial \mathbb{B}_r(\zeta)$. In the present situation, this formula is applied for $f(s) = x^{-s}$.

as a sum

$$v(x) = v^0(x) + \sum_{\lambda \in \Lambda} \sum_{j=0}^N x^{-\lambda} (\log x)^j v_{\lambda,j}(x),$$

with $\Lambda \subset \mathbb{C}$ a finite set such that $\lambda \not\sim_1 \lambda'$ for any two distinct points $\lambda, \lambda' \in \Lambda$, $N \in \mathbb{N}$, $v^0 \in \mathcal{S}(\mathbb{R}_+)$ and $v_{\lambda,j} \in \mathcal{S}(\overline{\mathbb{R}}_+)$. This expression is unique up to addition of Schwartz functions in $\mathcal{S}(\mathbb{R}_+)$.

On the other hand, we define the space \mathcal{M}_{sis} of all meromorphic functions H satisfying the following conditions,

- the poles of H lie in a finite number of sets $\lambda - \mathbb{N}$,
- the orders of the poles of H are uniformly bounded, and
- H decays rapidly along vertical lines, uniformly on vertical strips.

Appealing to certain Gamma identities for the Gamma function in [MS3, Lemma 6.24], one may show, in the same way as [MS3, Lemma 6.35], that

$$\mathcal{M}_{\text{sis}} = \sum_{\lambda \in \mathbb{C}} \sum_{j \in \mathbb{N}} \mathcal{M}_{\text{sis}}^{\lambda,j}.$$

We have $\mathcal{M}_{\text{sis}}^{\lambda,j} \subseteq \mathcal{M}_{\text{sis}}^{\lambda',j}$ if and only if $\lambda \leq_1 \lambda'$, and $\mathcal{M}_{\text{sis}}^{\lambda,j} \cap \mathcal{M}_{\text{sis}}^{\lambda',j'} = \mathcal{H}_{\text{rd}}$ if either $j \neq j'$ or $\lambda \not\sim \lambda'$. Therefore

$$(3.7) \quad \mathcal{M}_{\text{sis}} / \mathcal{H}_{\text{rd}} = \bigoplus_{\omega \in \mathbb{C} / \sim_1} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \mathcal{M}_{\text{sis}}^{\lambda,j} / \mathcal{H}_{\text{rd}}.$$

The following lemma is a direct consequence of Lemma 3.1.

LEMMA 3.2. *The Mellin transform \mathcal{M} is an isomorphism between $\mathcal{S}_{\text{sis}}(\mathbb{R}_+)$ and \mathcal{M}_{sis} that respects their decompositions (3.6) and (3.7).*

3.1.3. More refined decompositions of $\mathcal{S}_{\text{sis}}(\mathbb{R}_+)$ and \mathcal{M}_{sis} . Alternatively, we define an order relation on \mathbb{C} , $\lambda \leq_2 \lambda'$ if $\lambda' - \lambda \in 2\mathbb{N}$, as well as an equivalence relation, $\lambda \sim_2 \lambda'$ if $\lambda' - \lambda \in 2\mathbb{Z}$.

Define $\mathcal{N}_{\text{sis}}^{\lambda,j}$ in the same way as $\mathcal{M}_{\text{sis}}^{\lambda,j}$ with $\lambda - \mathbb{N}$ replaced by $\lambda - 2\mathbb{N}$. Under the isomorphism via \mathcal{M} in Lemma 3.1, $\mathcal{N}_{\text{sis}}^{\lambda,j}$ is then isomorphic to $x^{-\lambda} (\log x)^j \mathcal{S}_0(\overline{\mathbb{R}}_+)$.

According to [MS3, Lemma 6.35], we have the following decomposition,

$$(3.8) \quad \mathcal{M}_{\text{sis}}^{\lambda,j} / \mathcal{H}_{\text{rd}} = \mathcal{N}_{\text{sis}}^{\lambda,j} / \mathcal{H}_{\text{rd}} \oplus \mathcal{N}_{\text{sis}}^{\lambda-1,j} / \mathcal{H}_{\text{rd}}.$$

Inserting this into (3.7), one obtains the following refined decomposition of $\mathcal{M}_{\text{sis}} / \mathcal{H}_{\text{rd}}$

$$\bigoplus_{\omega \in \mathbb{C} / \sim_1} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \left(\mathcal{N}_{\text{sis}}^{\lambda,j} / \mathcal{H}_{\text{rd}} \oplus \mathcal{N}_{\text{sis}}^{\lambda-1,j} / \mathcal{H}_{\text{rd}} \right) = \bigoplus_{\omega \in \mathbb{C} / \sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \mathcal{N}_{\text{sis}}^{\lambda,j} / \mathcal{H}_{\text{rd}}.$$

Under the isomorphism via \mathcal{M} in Lemma 3.2, the reflection of this refinement on the decomposition of $\mathcal{S}_{\text{sis}}(\mathbb{R}_+) / \mathcal{S}(\mathbb{R}_+)$ is

$$\bigoplus_{\omega \in \mathbb{C} / \sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \left(x^{-\lambda} (\log x)^j \mathcal{S}_0(\overline{\mathbb{R}}_+) \right) / \mathcal{S}(\mathbb{R}_+).$$

LEMMA 3.3. *We have the following refinements of the decompositions (3.6, 3.7),*

$$(3.9) \quad \mathcal{S}_{\text{sis}}(\mathbb{R}_+)/\mathcal{S}(\mathbb{R}_+) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} (x^{-\lambda}(\log x)^j \mathcal{S}_0(\overline{\mathbb{R}}_+)) / \mathcal{S}(\mathbb{R}_+).$$

$$(3.10) \quad \mathcal{M}_{\text{sis}}/\mathcal{H}_{\text{rd}} = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \mathcal{N}_{\text{sis}}^{\lambda, j} / \mathcal{H}_{\text{rd}}.$$

The Mellin transform \mathcal{M} respects these two decompositions.

COROLLARY 3.4. *Let $\delta \in \mathbb{Z}/2\mathbb{Z}$ and $m \in \mathbb{Z}$, and recall the definitions of $\mathcal{S}_\delta(\overline{\mathbb{R}}_+)$ and $\mathcal{S}_m(\overline{\mathbb{R}}_+)$ in §2.4.1.*

(1). *The Mellin transform \mathcal{M} respects the following decompositions,*

$$(3.11) \quad \mathcal{S}_{\text{sis}}(\mathbb{R}_+)/\mathcal{S}(\mathbb{R}_+) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} (x^{-\lambda}(\log x)^j \mathcal{S}_\delta(\overline{\mathbb{R}}_+)) / \mathcal{S}(\mathbb{R}_+),$$

$$(3.12) \quad \mathcal{M}_{\text{sis}}/\mathcal{H}_{\text{rd}} = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \mathcal{N}_{\text{sis}}^{\lambda-\delta, j} / \mathcal{H}_{\text{rd}}.$$

(2). *The Mellin transform \mathcal{M} respects the following decompositions,*

$$(3.13) \quad \mathcal{S}_{\text{sis}}(\mathbb{R}_+)/\mathcal{S}(\mathbb{R}_+) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} (x^{-\lambda}(\log x)^j \mathcal{S}_m(\overline{\mathbb{R}}_+)) / \mathcal{S}(\mathbb{R}_+),$$

$$(3.14) \quad \mathcal{M}_{\text{sis}}/\mathcal{H}_{\text{rd}} = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \mathcal{N}_{\text{sis}}^{\lambda-|m|, j} / \mathcal{H}_{\text{rd}}.$$

PROOF. These follow from Lemma 3.3 in conjunction with $x^\delta \mathcal{S}_0(\overline{\mathbb{R}}_+) = \mathcal{S}_\delta(\overline{\mathbb{R}}_+)$ and $x^{|m|} \mathcal{S}_0(\overline{\mathbb{R}}_+) = \mathcal{S}_m(\overline{\mathbb{R}}_+)$. Q.E.D.

REMARK 3.5. *Set $\lambda = 0$ and $j = 0$ in (3.8). It follows from the isomorphism \mathcal{M} the decomposition as below,*

$$\mathcal{S}(\overline{\mathbb{R}}_+)/\mathcal{S}(\mathbb{R}_+) = \mathcal{S}_0(\overline{\mathbb{R}}_+)/\mathcal{S}(\mathbb{R}_+) \oplus x \mathcal{S}_0(\overline{\mathbb{R}}_+)/\mathcal{S}(\mathbb{R}_+).$$

Since $x \mathcal{S}_0(\overline{\mathbb{R}}_+) = \mathcal{S}_1(\overline{\mathbb{R}}_+)$, one obtains $\mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}_0(\overline{\mathbb{R}}_+) + \mathcal{S}_1(\overline{\mathbb{R}}_+)$ and therefore $C^\infty(\overline{\mathbb{R}}_+) = C_0^\infty(\overline{\mathbb{R}}_+) + C_1^\infty(\overline{\mathbb{R}}_+)$. See Remark 2.5.

3.2. The spaces $\mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$ and $\mathcal{M}_{\text{sis}}^{\mathbb{R}}$. Following [MS3, (6.10)], we write $(\lambda, \delta) \leq (\lambda', \delta')$ if $\lambda' - \lambda \in \mathbb{N}$ and $\lambda' - \lambda \equiv \delta' + \delta \pmod{2}$ and $(\lambda, \delta) \sim (\lambda', \delta')$ if $\lambda' - \lambda - (\delta' + \delta) \in 2\mathbb{Z}$. Again, these define an order relation and an equivalence relation on $\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$.

3.2.1. *The space $\mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$.* According to [MS3, Definition 6.4] and [MS3, Lemma 6.35], define

$$\mathcal{S}_{\text{sis}}(\mathbb{R}^\times) = \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \sum_{\lambda \in \mathbb{C}} \sum_{j \in \mathbb{N}} \text{sgn}(x)^\delta |x|^{-\lambda} (\log |x|)^j \mathcal{S}(\mathbb{R}).$$

We have the following decomposition,

$$\mathcal{S}_{\text{sis}}(\mathbb{R}^\times)/\mathcal{S}(\mathbb{R}^\times) = \bigoplus_{\omega \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}/\sim} \bigoplus_{j \in \mathbb{N}} \varinjlim_{(\lambda, \delta) \in \omega} (\text{sgn}(x)^\delta |x|^{-\lambda} (\log |x|)^j \mathcal{S}(\mathbb{R})) / \mathcal{S}(\mathbb{R}^\times).$$

It follows from $\text{sgn}(x)|x|\mathcal{S}(\mathbb{R}) = x\mathcal{S}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ that

$$\mathcal{S}_{\text{sis}}(\mathbb{R}^\times)/\mathcal{S}(\mathbb{R}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} (|x|^{-\lambda} (\log |x|)^j \mathcal{S}(\mathbb{R})) / \mathcal{S}(\mathbb{R}^\times).$$

We let $\mathcal{S}_{\text{sis}}^\delta(\mathbb{R}^\times)$ denote the space of functions $v \in \mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$ satisfying the parity condition (2.17). Clearly, $\mathcal{S}_{\text{sis}}(\mathbb{R}^\times) = \mathcal{S}_{\text{sis}}^0(\mathbb{R}^\times) \oplus \mathcal{S}_{\text{sis}}^1(\mathbb{R}^\times)$. Then,

$$\mathcal{S}_{\text{sis}}^\delta(\mathbb{R}^\times)/\mathcal{S}_\delta(\mathbb{R}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} (|x|^{-\lambda} (\log |x|)^j \mathcal{S}_\delta(\mathbb{R})) / \mathcal{S}_\delta(\mathbb{R}^\times),$$

where $\mathcal{S}_\delta(\mathbb{R})$ and $\mathcal{S}_\delta(\mathbb{R}^\times)$ are defined in §2.4.1 and §2.4.2 respectively. Since $\mathcal{S}_\delta(\mathbb{R}) = \text{sgn}(x)^\delta \mathcal{S}_\delta(\overline{\mathbb{R}}_+)$,

$$(3.15) \quad \mathcal{S}_{\text{sis}}^\delta(\mathbb{R}^\times)/\mathcal{S}_\delta(\mathbb{R}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} (\text{sgn}(x)^\delta |x|^{-\lambda} (\log |x|)^j \mathcal{S}_\delta(\overline{\mathbb{R}}_+)) / \mathcal{S}_\delta(\mathbb{R}^\times).$$

Then, $|x|^{-\delta} \mathcal{S}_\delta(\overline{\mathbb{R}}_+) = \mathcal{S}_0(\overline{\mathbb{R}}_+)$ together with $\mathcal{S}_0(\overline{\mathbb{R}}_+)/\mathcal{S}(\mathbb{R}_+) \oplus |x|\mathcal{S}_0(\overline{\mathbb{R}}_+)/\mathcal{S}(\mathbb{R}_+) = \mathcal{S}(\overline{\mathbb{R}}_+)/\mathcal{S}(\mathbb{R}_+)$ (see Remark 3.5) yields

$$(3.16) \quad \mathcal{S}_{\text{sis}}^\delta(\mathbb{R}^\times)/\mathcal{S}_\delta(\mathbb{R}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_1} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} (\text{sgn}(x)^\delta |x|^{-\lambda} (\log |x|)^j \mathcal{S}(\overline{\mathbb{R}}_+)) / \mathcal{S}_\delta(\mathbb{R}^\times).$$

In particular,

$$\mathcal{S}_{\text{sis}}^\delta(\mathbb{R}^\times) = \text{sgn}(x)^\delta \mathcal{S}_{\text{sis}}(\mathbb{R}_+) = \{ \text{sgn}(x)^\delta v(|x|) : v \in \mathcal{S}_{\text{sis}}(\mathbb{R}_+) \}.$$

3.2.2. The space $\mathcal{M}_{\text{sis}}^\mathbb{R}$. We simply define $\mathcal{M}_{\text{sis}}^\mathbb{R} = \mathcal{M}_{\text{sis}} \times \mathcal{M}_{\text{sis}}$.

3.2.3. Isomorphism between $\mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$ and $\mathcal{M}_{\text{sis}}^\mathbb{R}$ via the Mellin transform $\mathcal{M}^\mathbb{R}$. Let $v \in \mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$. Since $v_\delta \in \mathcal{S}_{\text{sis}}(\overline{\mathbb{R}}_+)$, the identity

$$\mathcal{M}_\delta v(s) = 2\mathcal{M}v_\delta(s)$$

extends the definition of the Mellin transform \mathcal{M}_δ onto the space $\mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$. Therefore, as a consequence of Lemma 3.1, Lemma 3.2 and Corollary 3.4 (1), the following lemma is readily established.

LEMMA 3.6. *For $\delta \in \mathbb{Z}/2\mathbb{Z}$, the Mellin transform \mathcal{M}_δ establishes an isomorphism between the spaces $\mathcal{S}_{\text{sis}}^\delta(\mathbb{R}^\times)$ and \mathcal{M}_{sis} that respects their decompositions (3.16) and (3.7) as well as (3.15) and (3.12). Therefore, $\mathcal{M}^\mathbb{R} = (\mathcal{M}_0, \mathcal{M}_1)$ establishes an isomorphism between $\mathcal{S}_{\text{sis}}(\mathbb{R}^\times) = \mathcal{S}_{\text{sis}}^0(\mathbb{R}^\times) \oplus \mathcal{S}_{\text{sis}}^1(\mathbb{R}^\times)$ and $\mathcal{M}_{\text{sis}}^\mathbb{R} = \mathcal{M}_{\text{sis}} \times \mathcal{M}_{\text{sis}}$.*

3.2.4. An alternative decomposition of $\mathcal{S}_{\text{sis}}^\delta(\mathbb{R}^\times)$. The following lemma follows from Corollary 3.4 (1) (compare [MS3, Corollary 6.17]).

LEMMA 3.7. *Let $\delta \in \mathbb{Z}/2\mathbb{Z}$. The Mellin transform \mathcal{M}_δ respects the following decompositions,*

$$(3.17) \quad \begin{aligned} & \mathcal{S}_{\text{sis}}^\delta(\mathbb{R}^\times)/\mathcal{S}_\delta(\mathbb{R}^\times) \\ &= \bigoplus_{\omega \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}/\sim} \bigoplus_{j \in \mathbb{N}} \varinjlim_{(\lambda, \epsilon) \in \omega} (\text{sgn}(x)^\epsilon |x|^{-\lambda} (\log |x|)^j \mathcal{S}_{\epsilon+\delta}(\mathbb{R})) / \mathcal{S}_\delta(\mathbb{R}^\times), \end{aligned}$$

$$(3.18) \quad \mathcal{M}_{\text{sis}}/\mathcal{H}_{\text{rd}} = \bigoplus_{\omega \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}/\sim} \bigoplus_{j \in \mathbb{N}} \varinjlim_{(\lambda, \epsilon) \in \omega} \mathcal{N}_{\text{sis}}^{\lambda - (\epsilon + \delta), j} / \mathcal{H}_{\text{rd}}.$$

3.3. The spaces $\mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$ and $\mathcal{M}_{\text{sis}}^\mathbb{C}$. We write $(\lambda, m) \leq (\lambda', m')$ if $\lambda' - \lambda \in |m' - m| + 2\mathbb{N}$ and $(\lambda, m) \sim (\lambda', m')$ if $\lambda' - \lambda - |m' - m| \in 2\mathbb{Z}$. These define an order relation and an equivalence relation on $\mathbb{C} \times \mathbb{Z}$.

3.3.1. *The space $\mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$.* In parallel to §3.2.1, we first define

$$\mathcal{S}_{\text{sis}}(\mathbb{C}^\times) = \sum_{m \in \mathbb{Z}} \sum_{\lambda \in \mathbb{C}} \sum_{j \in \mathbb{N}} [z]^{-m} |z|^{-\lambda} (\log |z|)^j \mathcal{S}(\mathbb{C}).$$

We have the following decomposition,

$$\mathcal{S}_{\text{sis}}(\mathbb{C}^\times)/\mathcal{S}(\mathbb{C}^\times) = \bigoplus_{\omega \in \mathbb{C} \times \mathbb{Z}/\sim} \bigoplus_{j \in \mathbb{N}} \varinjlim_{(\lambda, m) \in \omega} ([z]^{-m} |z|^{-\lambda} (\log |z|)^j \mathcal{S}(\mathbb{C})) / \mathcal{S}(\mathbb{C}^\times).$$

It follows from $[z]|z|\mathcal{S}(\mathbb{C}) = z\mathcal{S}(\mathbb{C}) \subset \mathcal{S}(\mathbb{C})$ that

$$(3.19) \quad \mathcal{S}_{\text{sis}}(\mathbb{C}^\times)/\mathcal{S}(\mathbb{C}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} (|z|^{-\lambda} (\log |z|)^j \mathcal{S}(\mathbb{C})) / \mathcal{S}(\mathbb{C}^\times).$$

We let $\mathcal{S}_{\text{sis}}^m(\mathbb{C}^\times)$ denote the space of functions $v \in \mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$ satisfying (2.21). Then,

$$(3.20) \quad \mathcal{S}_{\text{sis}}^m(\mathbb{C}^\times)/\mathcal{S}_m(\mathbb{C}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} (|z|^{-\lambda} (\log |z|)^j \mathcal{S}_m(\mathbb{C})) / \mathcal{S}_m(\mathbb{C}^\times),$$

where $\mathcal{S}_m(\mathbb{C})$ and $\mathcal{S}_m(\mathbb{C}^\times)$ are defined in §2.4.1 and §2.4.2 respectively. Since $\mathcal{S}_m(\mathbb{C}) = [z]^m \mathcal{S}_m(\overline{\mathbb{R}}_+)$,

$$\mathcal{S}_{\text{sis}}^m(\mathbb{C}^\times)/\mathcal{S}_m(\mathbb{C}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} ([z]^m |z|^{-\lambda} (\log |z|)^j \mathcal{S}_m(\overline{\mathbb{R}}_+)) / \mathcal{S}_m(\mathbb{C}^\times).$$

Then, $|z|^{-|m|} \mathcal{S}_m(\overline{\mathbb{R}}_+) = \mathcal{S}_0(\overline{\mathbb{R}}_+)$ together with $\mathcal{S}_0(\overline{\mathbb{R}}_+)/\mathcal{S}(\mathbb{R}_+) \oplus |z| \mathcal{S}_0(\overline{\mathbb{R}}_+)/\mathcal{S}(\mathbb{R}_+) = \mathcal{S}(\overline{\mathbb{R}}_+)/\mathcal{S}(\mathbb{R}_+)$ yields

$$(3.21) \quad \mathcal{S}_{\text{sis}}^m(\mathbb{C}^\times)/\mathcal{S}_m(\mathbb{C}^\times) = \bigoplus_{\omega \in \mathbb{C}/\sim_1} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} ([z]^m |z|^{-\lambda} (\log |z|)^j \mathcal{S}(\overline{\mathbb{R}}_+)) / \mathcal{S}_m(\mathbb{C}^\times).$$

In particular,

$$\mathcal{S}_{\text{sis}}^m(\mathbb{C}^\times) = [z]^m \mathcal{S}_{\text{sis}}(\mathbb{R}_+) = \{[z]^m v(|z|) : v \in \mathcal{S}_{\text{sis}}(\mathbb{R}_+)\}.$$

3.3.2. *The space $\mathcal{M}_{\text{sis}}^\mathbb{C}$.* For $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$, we define the space $\mathcal{N}_{\text{sis}}^{\mathbb{C}, \lambda, j}$ of all sequences $\{H_m(s)\}_{m \in \mathbb{Z}}$ of meromorphic functions such that

- the only singularities of H_m are poles of pure order $j + 1$ at the points in $\lambda - |m| - 2\mathbb{N}$,
- Each H_m decays rapidly along vertical lines, uniformly on vertical strips (see (3.1)), and
- $H_m(s)$ also decays rapidly with respect to m , uniformly on vertical strips, in the sense that

$$(3.22) \quad \text{for any given } \alpha, A \in \mathbb{N} \text{ and vertical strip } \mathbb{S}[a, b],$$

$$H_m(s) \ll_{\lambda, j, \alpha, A, a, b} (|m| + 1)^{-A} (|\Im m| + 1)^{-\alpha} \quad \text{for all } s \in \mathbb{S}[a, b], \text{ if } |m| > \Re \lambda - a.$$

Observe that the first two conditions amount to $H_m \in \mathcal{N}_{\text{sis}}^{\lambda-|m|,j}$. Therefore, $\mathcal{N}_{\text{sis}}^{\mathbb{C},\lambda,j} \subset \prod_{m \in \mathbb{Z}} \mathcal{N}_{\text{sis}}^{\lambda-|m|,j}$.

Define the space $\mathcal{M}_{\text{sis}}^{\mathbb{C}}$ of all sequences $\{H_m\}_{m \in \mathbb{Z}}$ of meromorphic functions such that

- the poles of each H_m lie in $\lambda - |m| - 2\mathbb{N}$, for a finite number of λ ,
- the orders of the poles of H_m are uniformly bounded,
- Each H_m decays rapidly along vertical lines, uniformly on vertical strips, and
- H_m decays rapidly with respect to m , uniformly on vertical strips.

Using the refined Stirling's asymptotic formula (2.13) in place of [MS3, (6.22)] and the following bound in place of [MS3, (6.23)]

$$\left| \frac{\Gamma^{(j)}\left(\frac{1}{2}(s - \lambda + |m|)\right)}{\Gamma\left(\frac{1}{2}(s + |m|)\right)} \right| \ll_{\lambda,j,a,b,r} (|\Im s| + |m| + 1)^{-\frac{1}{2}\Re \lambda}$$

for $\lambda \in \mathbb{C}$, $j \in \mathbb{N}$, $s \in \mathbb{S}[a, b] \setminus \bigcup_{\substack{\kappa \geq |m| \\ \kappa \equiv m \pmod{2}}} \mathbb{B}_r(\lambda - \kappa)$, with $r > 0$, we may follow the same lines of the proofs of [MS3, Lemma 6.24] and [MS3, Lemma 6.35] to show that

$$\mathcal{M}_{\text{sis}}^{\mathbb{C}} = \sum_{\lambda \in \mathbb{C}} \sum_{j \in \mathbb{N}} \mathcal{N}_{\text{sis}}^{\mathbb{C},\lambda,j}.$$

Then, we have

$$(3.23) \quad \mathcal{M}_{\text{sis}}^{\mathbb{C}} / \mathcal{H}_{\text{rd}}^{\mathbb{C}} = \bigoplus_{\omega \in \mathbb{C}/\sim_2} \bigoplus_{j \in \mathbb{N}} \varinjlim_{\lambda \in \omega} \mathcal{N}_{\text{sis}}^{\mathbb{C},\lambda,j} / \mathcal{H}_{\text{rd}}^{\mathbb{C}}.$$

3.3.3. Isomorphism between $\mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$ and $\mathcal{M}_{\text{sis}}^{\mathbb{C}}$ via the Mellin transform $\mathcal{M}^{\mathbb{C}}$. For $v \in \mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$, its m -th Fourier coefficient v_m is a function in $\mathcal{S}_{\text{sis}}(\overline{\mathbb{R}}_+)$. Hence the identity

$$\mathcal{M}_{-m}v(s) = 4\pi\mathcal{M}v_m(s)$$

extends the definition of the Mellin transform \mathcal{M}_{-m} onto the space $\mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$.

LEMMA 3.8. *For $m \in \mathbb{Z}$, the Mellin transform \mathcal{M}_{-m} establishes an isomorphism between the spaces $\mathcal{S}_{\text{sis}}^m(\mathbb{C}^\times)$ and \mathcal{M}_{sis} that respects their decompositions (3.21) and (3.7) as well as (3.20) and (3.14). Furthermore, $\mathcal{M}_{\mathbb{C}} = \prod_{m \in \mathbb{Z}} \mathcal{M}_{-m}$ establishes an isomorphism between $|z|^{-\lambda}(\log |z|)^j \mathcal{S}(\mathbb{C})$ and $\mathcal{N}_{\text{sis}}^{\mathbb{C},\lambda,j}$ for any $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$, and hence an isomorphism between $\mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$ and $\mathcal{M}_{\text{sis}}^{\mathbb{C}}$ that respects their decompositions (3.19) and (3.23).*

PROOF. For $v \in \mathcal{S}_{\text{sis}}^m(\mathbb{C}^\times)$, one has $v(xe^{i\phi}) = e^{im\phi}v_m(x)$ and $v_m \in \mathcal{S}_{\text{sis}}(\overline{\mathbb{R}}_+)$. Thus the first assertion follows immediately from Lemma 3.2 and Corollary 3.4 (2).

Now let $\varphi \in \mathcal{S}(\mathbb{C})$ and $v(z) = |z|^{-\lambda}(\log |z|)^j \varphi(z)$. Clearly, their m -th Fourier coefficients are related by $v_m(x) = x^{-\lambda}(\log x)^j \varphi_m(x)$. Since $\varphi_m \in \mathcal{S}_m(\overline{\mathbb{R}}_+)$, it follows from Corollary 3.4 (2) that $H_m = \mathcal{M}_{-m}v = 4\pi\mathcal{M}v_m$ lies in $\mathcal{N}_{\text{sis}}^{\lambda-|m|,j}$, and therefore we are left to show (3.22). Recall that in the proof of Lemma 3.1 we turned to verify (3.2) instead of (3.1). Likewise, it is more convenient to verify the following equivalent statement of (3.22),

$$(3.24) \quad \text{for any given } \alpha, A \in \mathbb{N}, b \geq a > \Re \lambda - \alpha - A - 1,$$

$$H_m(s) \ll_{\lambda,j,\alpha,A,b} (|m| + 1)^{-A} (|\Im s| + 1)^{-\alpha} \text{ for all } s \in \mathbb{S}[a, b], \text{ if } |m| > \alpha + A.$$

According to Lemma 2.4 (3.1), φ_m satisfies the conditions (2.32, 2.35). Suppose $|m| > \alpha + A$. One directly applies (2.32) and (2.35) to bound the following integral by a constant multiple of $(|m| + 1)^{-A}$,

$$(-)^\alpha (s - \lambda)_\alpha \mathcal{M}v_m(s) = \int_0^\infty \frac{d^\alpha}{dx^\alpha} ((\log x)^j \varphi_m(x)) x^{s-\lambda+\alpha-1} dx.$$

This proves (3.24) for $H_m = 4\pi \mathcal{M}v_m$. Therefore, the sequence $\{\mathcal{M}_{-m}v\}_{m \in \mathbb{Z}}$ belongs to $\mathcal{N}_{\text{sis}}^{\mathbb{C}, \lambda, j}$.

Conversely, let $\{H_m\}_{m \in \mathbb{Z}} \in \mathcal{N}_{\text{sis}}^{\mathbb{C}, \lambda, j}$, and let $4\pi v_m$ be the Mellin inversion of H_m ,

$$v_m(x) = \frac{1}{8\pi^2 i} \int_{(\sigma)} H_m(s) x^{-s} ds, \quad \sigma > \Re \lambda - |m|.$$

Since $H_m \in \mathcal{N}_{\text{sis}}^{\lambda-|m|, j}$, Corollary 3.4 (2) implies that $v_m(x) \in x^{-\lambda} (\log x)^j \mathcal{S}_m(\overline{\mathbb{R}}_+)$ and hence $\varphi_m(x) = x^\lambda (\log x)^{-j} v_m(x)$ lies in $\mathcal{S}_m(\overline{\mathbb{R}}_+)$. This proves (2.34). Similar to the proof of Lemma 3.1, right shifting of the contour of integration combined with (3.24) yields (2.32), whereas left shifting combined with (3.24) yields (2.35)^{VII}.

The proof of the second assertion is completed.

Q.E.D.

3.3.4. *An alternative decomposition of $\mathcal{S}_{\text{sis}}^m(\mathbb{C}^\times)$.* The following lemma follows from Corollary 3.4 (2).

LEMMA 3.9. *Let $m \in \mathbb{Z}$. The Mellin transform \mathcal{M}_{-m} respects the following decompositions,*

$$(3.25) \quad \begin{aligned} & \mathcal{S}_{\text{sis}}^m(\mathbb{C}^\times) / \mathcal{S}_m(\mathbb{C}^\times) \\ &= \bigoplus_{\omega \in \mathbb{C} \times \mathbb{Z} / \sim} \bigoplus_{j \in \mathbb{N}} \varinjlim_{(\lambda, k) \in \omega} ([z]^{-k} |z|^{-\lambda} (\log |z|)^j \mathcal{S}_{m+k}(\mathbb{C})) / \mathcal{S}_m(\mathbb{C}^\times), \end{aligned}$$

$$(3.26) \quad \mathcal{M}_{\text{sis}} / \mathcal{H}_{\text{rd}} = \bigoplus_{\omega \in \mathbb{C} \times \mathbb{Z} / \sim} \bigoplus_{j \in \mathbb{N}} \varinjlim_{(\lambda, k) \in \omega} \mathcal{N}_{\text{sis}}^{\lambda-|m+k|, j} / \mathcal{H}_{\text{rd}}.$$

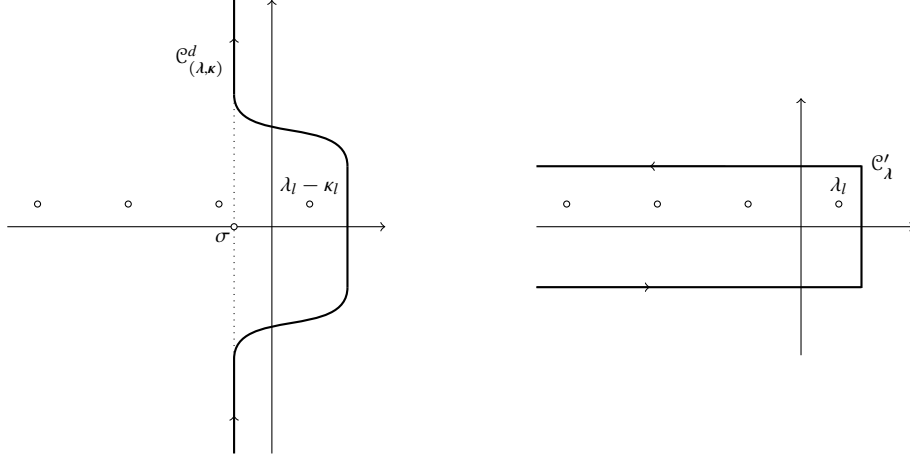
4. Hankel transforms and their Bessel kernels

This section is arranged as follows. We start with the type of Hankel transforms over \mathbb{R}_+ whose kernels are the (fundamental) Bessel functions studied in [Qi1]. After this, we introduce two auxiliary Hankel transforms and Bessel kernels over \mathbb{R}_+ . Finally, we proceed to construct and study Hankel transforms and their Bessel kernels over \mathbb{F}^\times , with $\mathbb{F} = \mathbb{R}, \mathbb{C}$.

DEFINITION 4.1. *Let (\mathbb{X}, \leq) be an ordered set satisfying the condition that*

$$(4.1) \quad “\lambda \leq \lambda' \text{ or } \lambda' \leq \lambda” \text{ is an equivalence relation.}$$

^{VII} Actually, $O_{\alpha, A}((|m| + 1)^{-A} x^{A+1})$ in (2.35) should be replaced by $O_{\alpha, A, \varrho}((|m| + 1)^{-A} x^{A+\varrho})$, $1 > \varrho > 0$. Moreover, one observes that the left contour shift here does not cross any pole.

FIGURE 1. $\mathcal{C}_{(\lambda, \kappa)}^d$ and \mathcal{C}'_λ

We denote the above equivalence relation by $\lambda \sim \lambda'$. Given $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{X}^n$, the set $\{1, \dots, n\}$ is partitioned into several pair-wise disjoint subsets L_α , $\alpha = 1, \dots, A$, such that

$$\lambda_l \sim \lambda_{l'} \text{ if and only if } l, l' \text{ are in the same } L_\alpha.$$

Each $\Lambda^\alpha = \{\lambda_l\}_{l \in L_\alpha}$ ^{VIII} is a totally ordered set. Let $B_\alpha = |\Lambda^\alpha|$ and label the elements of Λ^α in the descending order, $\lambda_{\alpha,1} > \dots > \lambda_{\alpha,B_\alpha}$. For $\lambda_{\alpha,\beta} \in \Lambda^\alpha$, let $M_{\alpha,\beta}$ denote the multiplicity of $\lambda_{\alpha,\beta}$ in λ , that is, $M_{\alpha,\beta} = |\{l : \lambda_l = \lambda_{\alpha,\beta}\}|$, and define $N_{\alpha,\beta} = \sum_{\gamma=1}^{\beta} M_{\alpha,\gamma} = |\{l : \lambda_{\alpha,\beta} \leq \lambda_l\}|$. λ is called generic if $\lambda_l \not\sim \lambda_{l'}$ for any $l \neq l'$.

We recall that the ordered sets (\mathbb{C}, \leq_1) , (\mathbb{C}, \leq_2) , $(\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}, \leq)$ and $(\mathbb{C} \times \mathbb{Z}, \leq)$ defined in §3 all satisfy (4.1).

DEFINITION 4.2. Let $d = 1$ or 2 , $\lambda \in \mathbb{C}^n$ and $\kappa \in \mathbb{N}^n$. Put $\sigma < \frac{d}{2} + \frac{1}{n}(\Re |\lambda| - 1)$ and choose a contour $\mathcal{C}_{(\lambda, \kappa)}^d$ (see Figure 1) such that

- $\mathcal{C}_{(\lambda, \kappa)}^d$ is upward directed from $\sigma - i\infty$ to $\sigma + i\infty$,
- all the sets $\lambda_l - \kappa_l - \mathbb{N}$ lie on the left hand side of $\mathcal{C}_{(\lambda, \kappa)}^d$, and
- if $s \in \mathcal{C}_{(\lambda, \kappa)}^d$ and $|\Im s|$ is sufficiently large, say $|\Im s| - \max\{|\Im \lambda_l|\} \gg 1$, then $\Re s = \sigma$.

For $\lambda \in \mathbb{C}$, we denote $\mathcal{C}_\lambda = \mathcal{C}_{(\lambda, \mathbf{0})}^1$. For $(\mu, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$, we denote $\mathcal{C}_{(\mu, \delta)} = \mathcal{C}_{(\mu, \delta)}^1$. For $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$, we denote $\mathcal{C}_{(\mu, m)} = \frac{1}{2} \cdot \mathcal{C}_{(2\mu, \|m\|)}^2$.

DEFINITION 4.3. For $\lambda \in \mathbb{C}^n$, choose a contour \mathcal{C}'_λ illustrated in Figure 1 such that

- \mathcal{C}'_λ starts from and returns to $-\infty$ counter-clockwise,
- \mathcal{C}'_λ consists two horizontal infinite half lines,
- \mathcal{C}'_λ encircles all the sets $\lambda_l - \mathbb{N}$, and

^{VIII}Here, $\{\lambda_l\}_{l \in L_\alpha}$ is considered as a set, namely, λ_l are counted without multiplicity.

- $\Im s \ll \max\{|\Im \lambda_l|\} + 1$ for all $s \in \mathcal{C}'_\lambda$.

4.1. The Hankel transform $\mathcal{H}_{(\mathfrak{s}, \lambda)}$ and the Bessel function $J(x; \mathfrak{s}, \lambda)$.

4.1.1. *The definition of $\mathcal{H}_{(\mathfrak{s}, \lambda)}$.* Consider the ordered set (\mathbb{C}, \leq_1) . For $\lambda \in \mathbb{C}^n$, let notations $\lambda_{\alpha\beta}$, B_α , $M_{\alpha\beta}$ and $N_{\alpha\beta}$ be as in Definition 4.1. We define the following subspace of $\mathcal{S}_{\text{sis}}(\mathbb{R}_+)$,

$$\mathcal{S}_{\text{sis}}^\lambda(\mathbb{R}_+) = \sum_{\alpha=1}^A \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha\beta}-1} x^{-\lambda_{\alpha\beta}} (\log x)^j \mathcal{S}(\overline{\mathbb{R}}_+).$$

PROPOSITION 4.4. *Let $(\mathfrak{s}, \lambda) \in \{+, -\}^n \times \mathbb{C}^n$. Suppose $v \in \mathcal{S}(\mathbb{R}_+)$. Then there exists a unique function $\Upsilon \in \mathcal{S}_{\text{sis}}^\lambda(\mathbb{R}_+)$ satisfying the following identity,*

$$(4.2) \quad \mathcal{M}\Upsilon(s) = G(s; \mathfrak{s}, \lambda) \mathcal{M}v(1-s).$$

We call Υ the Hankel transform of v over \mathbb{R}_+ of index (\mathfrak{s}, λ) and write $\mathcal{H}_{(\mathfrak{s}, \lambda)}v = \Upsilon$.

PROOF. Recall the definition of $G(s; \mathfrak{s}, \lambda)$ given by (2.1, 2.2),

$$G(s; \mathfrak{s}, \lambda) = e \left(\frac{\sum_{l=1}^n s_l(s - \lambda_l)}{4} \right) \prod_{l=1}^n \Gamma(s - \lambda_l).$$

The product in the above expression may be rewritten as below

$$\prod_{\alpha=1}^A \prod_{\beta=1}^{B_\alpha} \Gamma(s - \lambda_{\alpha\beta})^{M_{\alpha\beta}}.$$

Thus the singularities of $G(s; \mathfrak{s}, \lambda)$ are poles at the points in $\lambda_{\alpha,1} - \mathbb{N}$, $\alpha = 1, \dots, A$. More precisely, $G(s; \mathfrak{s}, \lambda)$ has a pole of pure order $N_{\alpha\beta}$ at $\lambda \in \lambda_{\alpha,1} - \mathbb{N}$ if one let $\beta = \max\{\beta' : \lambda \leq_1 \lambda_{\alpha\beta'}\}$. Moreover, in view of (2.14) in Lemma 2.2, $G(s; \mathfrak{s}, \lambda)$ is of uniform moderate growth on vertical strips.

On the other hand, according to Corollary 2.9 (1), $\mathcal{M}v(1-s)$ uniformly rapidly decays on vertical strips.

Therefore, the product $G(s; \mathfrak{s}, \lambda) \mathcal{M}v(1-s)$ on the right hand side of (4.2) is a meromorphic function in the space $\sum_{\alpha=1}^A \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha\beta}-1} \mathcal{M}_{\text{sis}}^{\lambda_{\alpha\beta}, j}$. We conclude from Lemma 3.2 that (4.2) uniquely determines a function Υ in $\mathcal{S}_{\text{sis}}^\lambda(\mathbb{R}_+)$. Q.E.D.

4.1.2. The Bessel function $J(x; \mathfrak{s}, \lambda)$.

The integral kernel $J(x; \mathfrak{s}, \lambda)$ of $\mathcal{H}_{(\mathfrak{s}, \lambda)}$. Suppose $v \in \mathcal{S}(\mathbb{R}_+)$. By the Mellin inversion, we have

$$(4.3) \quad \Upsilon(x) = \frac{1}{2\pi i} \int_{(\sigma)} G(s; \mathfrak{s}, \lambda) \mathcal{M}v(1-s) x^{-s} ds, \quad \sigma > \max\{\Re \lambda_l\}.$$

It is an iterated double integral as below

$$\Upsilon(x) = \frac{1}{2\pi i} \int_{(\sigma)} \int_0^\infty v(y) y^{-s} dy \cdot G(s; \mathfrak{s}, \lambda) x^{-s} ds.$$

We now shift the integral contour to \mathcal{C}_λ defined in Definition 4.2. Using (2.14) in Lemma 2.2, one shows that the above double integral becomes absolutely convergent after this contour shift. Therefore, on changing the order of integrals, one obtains

$$(4.4) \quad \Upsilon(x) = \int_0^\infty v(y) J((xy)^\frac{1}{n}; \mathfrak{s}, \lambda) dy.$$

Here $J(x; \mathfrak{s}, \lambda)$ is the (fundamental) Bessel function defined by the Barnes-Mellin type integral

$$(4.5) \quad J(x; \mathfrak{s}, \lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}_\lambda} G(s; \mathfrak{s}, \lambda) x^{-ns} ds,$$

which is categorized as a *Bessel function of the second kind* (see [Qi1]).

REMARK 4.5. The expression (4.4) of the Hankel transform together with properties of the Bessel function $J(x; \mathfrak{s}, \lambda)$ may also yield $\Upsilon \in \mathcal{S}_{\text{sis}}^\lambda(\mathbb{R}_+)$.

The Schwartz condition on Υ at infinity follows from either the rapid decay or the oscillation of $J(x; \mathfrak{s}, \lambda)$ as well as its derivatives (see [Qi1, §5, §9]).

As for the singularity type of Υ at zero, we first assume that λ is generic. We express $J(x; \mathfrak{s}, \lambda)$ as a combination of Bessel functions of the first kind (see [Qi1, §7.1, 7.2]). Then the type of singularities of Υ at zero is reflected by the leading term in the series expansions of Bessel functions of the first kind. For nongeneric λ the occurrence of powers of $\log x$ follows from either solving the Bessel equations using the Frobenius method or taking the limit of the above expression of $J(x; \mathfrak{s}, \lambda)$ with respect to the index λ .

Shifting the index of $J(x; \mathfrak{s}, \lambda)$.

LEMMA 4.6. Let $(\mathfrak{s}, \lambda) \in \{+, -\}^n \times \mathbb{C}^n$ and $\lambda \in \mathbb{C}$. Recall that \mathbf{e}^n denotes the n -tuple $(1, \dots, 1)$. Then

$$(4.6) \quad J(x; \mathfrak{s}, \lambda - \lambda \mathbf{e}^n) = x^{n\lambda} J(x; \mathfrak{s}, \lambda).$$

Regularity of $J(x; \mathfrak{s}, \lambda)$. According to [Qi1, §6, 7], $J(x; \mathfrak{s}, \lambda)$ satisfies a differential equation with analytic coefficients. Therefore, $J(x; \mathfrak{s}, \lambda)$ admits an analytic continuation from \mathbb{R}_+ onto \mathbb{U} , and in particular is *real analytic*. Here, we shall take an alternative viewpoint from [Qi1, Remark 7.11], that is the following Barnes type integral representation,

$$(4.7) \quad J(\zeta; \mathfrak{s}, \lambda) = \frac{1}{2\pi i} \int_{\mathcal{C}'_\lambda} G(s; \mathfrak{s}, \lambda) \zeta^{-ns} ds, \quad \zeta = xe^{i\omega} \in \mathbb{U}, x \in \mathbb{R}_+, \omega \in \mathbb{R},$$

with the integral contour given in Definition 4.3. One first rewrites $G_\pm(s)$ using Euler's reflection formula,

$$G_\pm(s) = \frac{\pi e\left(\pm\frac{1}{4}s\right)}{\sin(\pi s)\Gamma(1-s)},$$

Then Stirling's formula (2.13) yields,

$$G(-\varrho + it; \mathfrak{s}, \lambda) \ll_{\lambda, r} e^{n\varrho} \varrho^{-n\left(\varrho + \frac{1}{2}\right) - \Re \varrho |\lambda|},$$

for all $-\varrho + it \notin \bigcup_{l=1}^n \bigcup_{\kappa \in \mathbb{N}} \mathbb{B}_r(\lambda_l - \kappa)$ satisfying $\varrho \gg 1$ and $t \ll \max\{|\Im \lambda_l|\} + 1$. It follows that the contour integral in (4.7) converges absolutely and compactly in ζ , and hence $J(\zeta; \mathbf{s}, \lambda)$ is analytic in ζ .

Moreover, given any bounded open subset of \mathbb{C}^n , one fixes a single contour $\mathcal{C}' = \mathcal{C}'_\lambda$ for all λ in this set and verifies the uniform convergence of the integral in the λ aspect. Then follows the analyticity of $J(\zeta; \mathbf{s}, \lambda)$ with respect to λ .

LEMMA 4.7. *$J(x; \mathbf{s}, \lambda)$ admits an analytic continuation $J(\zeta; \mathbf{s}, \lambda)$ from \mathbb{R}_+ onto \mathbb{U} . In particular, $J(x; \mathbf{s}, \lambda)$ is a real analytic function of x on \mathbb{R}_+ . Moreover, $J(\zeta; \mathbf{s}, \lambda)$ is an analytic function of λ on \mathbb{C}^n .*

The rank-one and rank-two cases.

EXAMPLE 4.8. *According to [Qi1, Proposition 2.4], if $n = 1$, then*

$$J(x; \pm, 0) = e^{\pm ix}.$$

For $n = 2$, from [Qi1, Proposition 2.7] we have

$$J(x; \pm, \pm, \lambda, -\lambda) = \pm \pi i e^{\pm \pi i \lambda} H_{2\lambda}^{(1,2)}(2x), \quad J(x; \pm, \mp, \lambda, -\lambda) = 2e^{\mp \pi i \lambda} K_{2\lambda}(2x),$$

where, for $\nu \in \mathbb{C}$, $H_\nu^{(1)}$, $H_\nu^{(2)}$ are the Hankel functions, and K_ν is the K -Bessel function (the modified Bessel function of the second kind).

4.2. The Hankel transforms $h_{(\mu, \delta)}$, $h_{(\mu, m)}$ and the Bessel kernels $j_{(\mu, \delta)}$, $j_{(\mu, m)}$. Consider the ordered set (\mathbb{C}, \leq_2) and define $\lambda_{\alpha, \beta}$, B_α , $M_{\alpha, \beta}$ and $N_{\alpha, \beta}$ as in Definition 4.1 corresponding to $\lambda \in \mathbb{C}^n$. We define the following subspace of $\mathcal{S}_{\text{sis}}(\mathbb{R}_+)$

$$(4.8) \quad \mathcal{S}_{\text{sis}}^\lambda(\mathbb{R}_+) = \sum_{\alpha=1}^A \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha, \beta}-1} x^{-\lambda_{\alpha, \beta}} (\log x)^j \mathcal{S}_0(\overline{\mathbb{R}}_+).$$

4.2.1. *The definition of $h_{(\mu, \delta)}$.* The following proposition provides the definition of the Hankel transform $h_{(\mu, \delta)}$, which maps $\mathcal{S}_{\text{sis}}^{-\mu-\delta}(\mathbb{R}_+)$ onto $\mathcal{S}_{\text{sis}}^{\mu-\delta}(\mathbb{R}_+)$ bijectively.

PROPOSITION 4.9. *Let $(\mu, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$. Suppose $v \in \mathcal{S}_{\text{sis}}^{-\mu-\delta}(\mathbb{R}_+)$. Then there exists a unique function $\Upsilon \in \mathcal{S}_{\text{sis}}^{\mu-\delta}(\mathbb{R}_+)$ satisfying the following identity,*

$$(4.9) \quad \mathcal{M}\Upsilon(s) = G_{(\mu, \delta)}(s) \mathcal{M}v(1-s).$$

We call Υ the Hankel transform of v over \mathbb{R}_+ of index (μ, δ) and write $h_{(\mu, \delta)}v = \Upsilon$. Furthermore, we have the Hankel inversion formula

$$(4.10) \quad h_{(\mu, \delta)}v = \Upsilon, \quad h_{(-\mu, \delta)}\Upsilon = v.$$

PROOF. Recall the definition of $G_{(\mu, \delta)}$ given by (2.3, 2.4),

$$G_{(\mu, \delta)}(s) = i^{|\delta|} \pi^{n(\frac{1}{2}-s)+|\mu|} \frac{\prod_{l=1}^n \Gamma(\frac{1}{2}(s - \mu_l + \delta_l))}{\prod_{l=1}^n \Gamma(\frac{1}{2}(1 - s + \mu_l + \delta_l))},$$

where $|\delta| = \sum_l \delta_l \in \mathbb{N}$, with each δ_l viewed as a number in the set $\{0, 1\} \subset \mathbb{N}$.

We write $\mu^\pm = \pm\mu - \delta$. Since $\mu_l^+ + \mu_l^- = -2\delta_l \in \{0, -2\}$, the partition $\{L_\alpha\}_{\alpha=1}^A$ of $\{1, \dots, n\}$ and B_α in Definition 4.1 are the same for both μ^+ and μ^- . Let $\mu_{\alpha,\beta}^\pm$, $M_{\alpha,\beta}^\pm$ and $N_{\alpha,\beta}^\pm$ be the notations in Definition 4.1 corresponding to μ^\pm . Then the Gamma quotient above may be rewritten as follows,

$$\frac{\prod_{\alpha=1}^A \prod_{\beta=1}^{B_\alpha} \Gamma\left(\frac{1}{2}(s - \mu_{\alpha,\beta}^+)\right)^{M_{\alpha,\beta}^+}}{\prod_{\alpha=1}^A \prod_{\beta=1}^{B_\alpha} \Gamma\left(\frac{1}{2}(1 - s - \mu_{\alpha,\beta}^-)\right)^{M_{\alpha,\beta}^-}}.$$

Thus, at each point $\mu \in \mu_{\alpha,1}^+ - 2\mathbb{N}$ the product in the numerator contributes to $G_{(\mu,\delta)}(s)$ a pole of pure order $N_{\alpha,\beta}^+$, with $\beta = \max\{\beta' : \mu \leq_2 \mu_{\alpha,\beta'}^+\}$, whereas at each point $\mu \in -\mu_{\alpha,1}^- + 2\mathbb{N} + 1$ the denominator contributes a zero of order $N_{\alpha,\beta}^-$, with $\beta = \max\{\beta' : 1 - \mu \leq_2 \mu_{\alpha,\beta'}^-\}$. Moreover, (2.15) in Lemma 2.2 implies that $G_{(\mu,\delta)}(s)$ is of uniform moderate growth on vertical strips.

According to Lemma 3.3, $\mathcal{M}\nu$ lies in the space $\sum_{\alpha=1}^A \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha,\beta}^- - 1} \mathcal{N}_{\text{sis}}^{\mu_{\alpha,\beta}^-, j}$. In particular, the poles of $\mathcal{M}\nu(1-s)$ are annihilated by the zeros contributed from the denominator of the Gamma quotient. Furthermore, $\mathcal{M}\nu(1-s)$ uniformly rapidly decays on vertical strips.

We conclude that the product $G_{(\mu,\delta)}(s)\mathcal{M}\nu(1-s)$ on the right hand side of (4.9) lies in the space $\sum_{\alpha=1}^A \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha,\beta}^+ - 1} \mathcal{N}_{\text{sis}}^{\mu_{\alpha,\beta}^+, j}$, and hence $\Upsilon \in \mathcal{T}_{\text{sis}}^{\mu-\delta}(\mathbb{R}_+)$, with another application of Lemma 3.3.

Finally, the Hankel inversion formula (4.10) is an immediate consequence of the functional relation (2.5) of gamma factors. Q.E.D.

4.2.2. The definition of $h_{(\mu,\mathbf{m})}$. The following proposition provides the definition of the Hankel transform $h_{(\mu,\mathbf{m})}$, which maps $\mathcal{T}_{\text{sis}}^{-2\mu-\|\mathbf{m}\|}(\mathbb{R}_+)$ onto $\mathcal{T}_{\text{sis}}^{2\mu-\|\mathbf{m}\|}(\mathbb{R}_+)$ bijectively.

PROPOSITION 4.10. *Let $(\mu, \mathbf{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$. Suppose $\nu \in \mathcal{T}_{\text{sis}}^{-2\mu-\|\mathbf{m}\|}(\mathbb{R}_+)$. Then there exists a unique function $\Upsilon \in \mathcal{T}_{\text{sis}}^{2\mu-\|\mathbf{m}\|}(\mathbb{R}_+)$ satisfying the following identity,*

$$(4.11) \quad \mathcal{M}\Upsilon(2s) = G_{(\mu,\mathbf{m})}(s)\mathcal{M}\nu(2(1-s)).$$

We call Υ the Hankel transform of ν over \mathbb{R}_+ of index (μ, \mathbf{m}) and write $h_{(\mu,\mathbf{m})}\nu = \Upsilon$. Moreover, we have the Hankel inversion formula

$$(4.12) \quad h_{(\mu,\mathbf{m})}\nu = \Upsilon, \quad h_{(-\mu,\mathbf{m})}\Upsilon = \nu.$$

PROOF. We first rewrite (4.11) as follows,

$$\mathcal{M}\Upsilon(s) = G_{(\mu,\mathbf{m})}\left(\frac{s}{2}\right)\mathcal{M}\nu(2-s).$$

From (2.6, 2.7), we have

$$G_{(\mu,\mathbf{m})}\left(\frac{s}{2}\right) = i^{\|\mathbf{m}\|} \pi^{n(1-s)+2|\mu|} \frac{\prod_{l=1}^n \Gamma\left(\frac{1}{2}(s - 2\mu_l + |m_l|)\right)}{\prod_{l=1}^n \Gamma\left(\frac{1}{2}(2 - s + 2\mu_l + |m_l|)\right)},$$

where $\|\mathbf{m}\| = \sum_{l=1}^n |m_l|$ according to our notations. We can now proceed to apply the same arguments in the proof of Proposition 4.9. Here, one uses (2.16) and (2.8) instead of (2.15) and (2.5) respectively. Q.E.D.

4.2.3. The Bessel kernel $j_{(\mu, \delta)}$.

The definition of $j_{(\mu, \delta)}$. For $(\mu, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$, we define the Bessel kernel $j_{(\mu, \delta)}$,

$$(4.13) \quad j_{(\mu, \delta)}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_{(\mu, \delta)}} G_{(\mu, \delta)}(s) x^{-s} ds.$$

We call the integral in (4.13) a Barnes-Mellin type integral. It is clear that

$$(4.14) \quad j_{(\mu - \mu e^n, \delta)}(x) = x^\mu j_{(\mu, \delta)}(x).$$

In view of (2.9), we have

$$(4.15) \quad j_{(\mu, \delta)}(x) = (2\pi)^{|\mu|} \sum_{\mathbf{s} \in \{+, -\}^n} \zeta^\delta J(2\pi x^{\frac{1}{n}}; \mathbf{s}, \mu).$$

Regularity of $j_{(\mu, \delta)}$. It follows from (4.15) and Lemma 4.7 that $j_{(\mu, \delta)}(x)$ admits an analytic continuation $j_{(\mu, \delta)}(\zeta)$, which is also analytic with respect to μ . Moreover, $j_{(\mu, \delta)}(\zeta)$ has the following Barnes type integral representation,

$$(4.16) \quad j_{(\mu, \delta)}(\zeta) = \frac{1}{2\pi i} \int_{\mathcal{C}'_{\mu - \delta}} G_{(\mu, \delta)}(s) \zeta^{-s} ds, \quad \zeta \in \mathbb{U}.$$

To see the convergence, the following formula is required

$$(4.17) \quad G_\delta(s) = \begin{cases} \frac{\pi(2\pi)^{-s}}{\sin(\frac{1}{2}\pi s) \Gamma(1-s)}, & \text{if } \delta = 0, \\ \frac{\pi i(2\pi)^{-s}}{\cos(\frac{1}{2}\pi s) \Gamma(1-s)}, & \text{if } \delta = 1. \end{cases}$$

The integral kernel of $h_{(\mu, \delta)}$. Suppose $v \in \mathcal{T}_{\text{sis}}^{-\mu - \delta}(\mathbb{R}_+)$. In order to proceed in the same way as in §4.1.2, one needs to assume that (μ, δ) satisfies the condition

$$(4.18) \quad \min \{\Re \mu_l + \delta_l\} + 1 > \max \{\Re \mu_l - \delta_l\}.$$

Then,

$$(4.19) \quad h_{(\mu, \delta)} v(x) = \int_0^\infty v(y) j_{(\mu, \delta)}(xy) dy.$$

Here, it is required for the convergence of the integral over dy that the contour $\mathcal{C}_{(\mu, \delta)}$ in (4.13) is chosen to lie in the left half-plane $\{s : \Re s < \min \{\Re \mu_l + \delta_l\} + 1\}$. According to Definition 4.2, this choice of $\mathcal{C}_{(\mu, \delta)}$ is permissible due to our assumption (4.18).

If one assumes $v \in \mathcal{S}(\mathbb{R}_+)$, then (4.19) remains valid without requiring the condition (4.18).

The rank-one and rank-two examples.

EXAMPLE 4.11. If $n = 1$, we have

$$(4.20) \quad j_{(0,0)}(x) = 2 \cos(2\pi x), \quad j_{(0,1)}(x) = 2i \sin(2\pi x).$$

If $n = 2$, we are particularly interested in the following Bessel kernel,

$$(4.21) \quad j_{(\frac{1}{2}m, -\frac{1}{2}m, \delta(m)+1, 0)}(x) = j_{(\frac{1}{2}m, -\frac{1}{2}m, \delta(m), 1)}(x) = 2\pi i^{m+1} J_m(4\pi \sqrt{x}),$$

with $m \in \mathbb{N}$. For $v \in \mathbb{C}$, J_v is the J -Bessel function (the Bessel function of the first kind).

4.2.4. The Bessel kernel $j_{(\mu, m)}$.

The definition of $j_{(\mu, m)}$. For $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$ define the Bessel kernel $j_{(\mu, m)}$ by

$$(4.22) \quad j_{(\mu, m)}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_{(\mu, m)}} G_{(\mu, m)}(s) x^{-2s} ds.$$

The integral in (4.22) is called a Barnes-Mellin type integral. We have

$$(4.23) \quad j_{(\mu - \mu e^n, m)}(x) = x^{2\mu} j_{(\mu, m)}(x).$$

In view of Lemma 2.1, if $(\eta, \delta) \in \mathbb{C}^{2n} \times (\mathbb{Z}/2\mathbb{Z})^{2n}$ is related to $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$ via either (2.11) or (2.12), then

$$(4.24) \quad i^n j_{(\mu, m)}(x) = j_{(\eta, \delta)}(x^2).$$

Regularity of $j_{(\mu, m)}$. In view of (4.24), the regularity of $j_{(\mu, m)}$ follows from that of $j_{(\eta, \delta)}$. Alternatively, this may be seen from

$$(4.25) \quad j_{(\mu, m)}(\zeta) = \frac{1}{2\pi i} \int_{\mathcal{C}'_{\mu - \frac{1}{2}\|m\|}} G_{(\mu, m)}(s) \zeta^{-2s} ds, \quad \zeta \in \mathbb{U}.$$

To see the convergence, the following formula is required

$$(4.26) \quad G_m(s) = \frac{\pi i^{|m|} (2\pi)^{1-2s}}{\sin(\pi(s + \frac{1}{2}|m|)) \Gamma(1-s - \frac{1}{2}|m|) \Gamma(1-s + \frac{1}{2}|m|)}.$$

The integral kernel of $h_{(\mu, m)}$. Suppose $v \in \mathcal{T}_{\text{sis}}^{-2\mu - \|m\|}(\mathbb{R}_+)$. We assume that (μ, m) satisfies the following condition

$$(4.27) \quad \min \left\{ \Re \mu_l + \frac{1}{2}|m_l| \right\} + 1 > \max \left\{ \Re \mu_l - \frac{1}{2}|m_l| \right\}.$$

Then

$$(4.28) \quad h_{(\mu, m)}v(x) = \int_0^\infty v(y) j_{(\mu, m)}(xy) \cdot 2y dy,$$

It is required for convergence that the integral contour $\mathcal{C}_{(\mu, m)}$ in (4.22) lies in the left half-plane $\{s : \Re s < \min \left\{ \Re \mu_l + \frac{1}{2}|m_l| \right\} + 1\}$. This is guaranteed by (4.27).

Moreover, if one assumes $v \in \mathcal{S}(\mathbb{R}_+)$, then (4.28) holds true for any index (μ, m) .

The rank-one case.

EXAMPLE 4.12. If $n = 1$, in view of (4.21) and (4.24), we have for $m \in \mathbb{Z}$

$$(4.29) \quad j_{(0,m)}(x) = 2\pi i^{|m|} J_{|m|}(4\pi x) = 2\pi i^m J_m(4\pi x).$$

where the second equality follows from the identity $J_{-m}(x) = (-)^m J_m(x)$.

Auxiliary bounds for $j_{(\mu, \mathbf{m} + m\mathbf{e}^n)}$.

LEMMA 4.13. Let $(\mu, \mathbf{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ and $m \in \mathbb{Z}$. Put

$$\begin{aligned} A &= n \left(\max\{\Re \mu_l\} + \frac{1}{2} \max\{|m_l|\} - \frac{1}{2} \right) - \Re |\mu| + \frac{1}{2} \|\mathbf{m}\|, \\ B_+ &= -2 \min\{\Re \mu_l\} + \max\{|m_l|\} + \max\left\{\frac{1}{n} - \frac{1}{2}, 0\right\}, \\ B_- &= -2 \max\{\Re \mu_l\} - \max\{|m_l|\}. \end{aligned}$$

Fix $\epsilon > 0$. Denote by \mathbf{e}^n the n -tuple $(1, \dots, 1)$. We have the following estimate

$$(4.30) \quad j_{(\mu, \mathbf{m} + m\mathbf{e}^n)}(x) \ll_{(\mu, \mathbf{m}), \epsilon, n} \left(\frac{2\pi e x^{\frac{1}{n}}}{|m| + 1} \right)^{n|m|} (|m| + 1)^{A+n\epsilon} \max\{x^{B_++2\epsilon}, x^{B_- - 2\epsilon}\}.$$

PROOF. Let

$$\begin{aligned} \rho_m &= \max\left\{\Re \mu_l - \frac{1}{2}|m_l + m|\right\}, \\ \sigma_m &= \min\left\{\frac{1}{2} + \frac{1}{n} \left(\Re |\mu| - \frac{1}{2} \|\mathbf{m} + m\mathbf{e}^n\| - 1\right), \rho_m\right\}. \end{aligned}$$

Choose the contour $\mathcal{C}_m = \mathcal{C}_{(\mu, \mathbf{m} + m\mathbf{e}^n)}$ (see Definition 4.2) such that

- if $s \in \mathcal{C}_m$ and $\Im s$ is sufficiently large, then $\Re s = \sigma_m - \epsilon$, and
- \mathcal{C}_m lies in the vertical strip $\mathbb{S}[\sigma_m - \epsilon, \rho_m + \epsilon]$.

We first assume that $|m|$ is large enough so that

$$n \left(\rho_m + \epsilon - \frac{1}{2} \right) - \Re |\mu| - \frac{1}{2} \|\mathbf{m} + m\mathbf{e}^n\| < 0.$$

For the sake of brevity, we write $y = (2\pi)^n x$. We first bound $|j_{(\mu, \mathbf{m} + m\mathbf{e}^n)}(x)|$ by

$$(2\pi)^{n+\Re |\mu|} \int_{\mathcal{C}_m} y^{-2\Re s} \prod_{l=1}^n \left| \frac{\Gamma(s - \mu_l + \frac{1}{2}|m_l + m|)}{\Gamma(1 - s + \mu_l + \frac{1}{2}|m_l + m|)} \right| |ds|.$$

With the observations that for $s \in \mathcal{C}_m$

- $\Re s \in [\sigma_m - \epsilon, \rho_m + \epsilon]$,
- $|\Re s - \mu_l + \frac{1}{2}|m_l + m|| \ll_{(\mu, \mathbf{m})} 1$,
- $|(1 - \Re s + \mu_l + \frac{1}{2}|m_l + m|) - |m|| \ll_{(\mu, \mathbf{m})} 1$,

in conjunction with Stirling's formula (2.13), one has the following estimate

$$\begin{aligned} j_{(\mu, \mathbf{m} + m\mathbf{e}^n)}(x) &\ll_{(\mu, \mathbf{m}), n, \epsilon} \max\{y^{-2\sigma_m+2\epsilon}, y^{-2\rho_m-2\epsilon}\} \\ &\quad \int_{\mathcal{C}_m} \frac{(|\Im s| + 1)^{n(\Re s - \frac{1}{2}) - \Re |\mu| + \frac{1}{2} \|\mathbf{m} + m\mathbf{e}^n\|}}{e^{-n|m|} (\sqrt{(\Im s)^2 + m^2} + 1)^{n(\frac{1}{2} - \Re s) + \Re |\mu| + \frac{1}{2} \|\mathbf{m} + m\mathbf{e}^n\|}} |ds| \\ &\leq \max\{y^{-2\sigma_m+2\epsilon}, y^{-2\rho_m-2\epsilon}\} e^{n|m|} (|m| + 1)^{n(\rho_m + \epsilon - \frac{1}{2}) - \Re |\mu| - \frac{1}{2} \|\mathbf{m} + m\mathbf{e}^n\|} \\ &\quad \int_{\mathcal{C}_m} (|\Im s| + 1)^{n(\Re s - \frac{1}{2}) - \Re |\mu| + \frac{1}{2} \|\mathbf{m} + m\mathbf{e}^n\|} |ds|. \end{aligned}$$

For $s \in \mathcal{C}_m$, one has $\Re s = \sigma_m - \epsilon$ if $\Im s$ is sufficiently large, and our choice of σ_m implies $n(\sigma_m - \epsilon - \frac{1}{2}) - \Re |\mu| + \frac{1}{2} \|\mathbf{m} + m\mathbf{e}^n\| \leq -1 - n\epsilon$, then it follows that the above integral converges and is of size $O_{(\mu, \mathbf{m}), n, \epsilon}(1)$.

Finally, note that both $-2\sigma_m + 2\epsilon$ and $-2\rho_m - 2\epsilon$ are close to $|m|$, whereas the exponent of $(|m| + 1)$, that is $n(\rho_m + \epsilon - \frac{1}{2}) - \Re |\mu| - \frac{1}{2} \|\mathbf{m} + m\mathbf{e}^n\|$, is close to $-n|m|$. Thus the following bounds yield (4.30),

$$\begin{aligned} |m| + B_- &\leq -2\rho_m \leq -2\sigma_m \leq |m| + B_+, \\ n(\rho_m - \frac{1}{2}) - \Re |\mu| - \frac{1}{2} \|\mathbf{m} + m\mathbf{e}^n\| &\leq -n|m| + A. \end{aligned}$$

When $|m|$ is small, we have the following estimate that also implies (4.30),

$$j_{(\mu, \mathbf{m} + m\mathbf{e}^n)}(x) \ll_{(\mu, \mathbf{m}), n, \epsilon} \max\{y^{-2\sigma_m + 2\epsilon}, y^{-2\rho_m - 2\epsilon}\} e^{n|m|}.$$

Q.E.D.

Using the formula (4.26) of $G_m(s)$ instead of (2.6) and the Barnes type integral representation (4.25) for $j_{(\mu, \mathbf{m} + m\mathbf{e}^n)}(\zeta)$ instead of the Barnes-Mellin type integral representation (4.22) for $j_{(\mu, \mathbf{m} + m\mathbf{e}^n)}(x)$, similar arguments in the proof of Lemma 4.13 imply the following lemma.

LEMMA 4.14. *Let $(\mu, \mathbf{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ and $m \in \mathbb{Z}$. Put*

$$\begin{aligned} A &= n(\max\{\Re \mu_l\} + \frac{1}{2} \max\{|m_l|\} - \frac{1}{2}) - \Re |\mu| + \frac{1}{2} \|\mathbf{m}\|, \\ B &= -2 \max\{\Re \mu_l\} - \max\{|m_l|\}, \quad C = 2 \max\{|\Im \mu_l|\}. \end{aligned}$$

Fix $X > 0$ and $\epsilon > 0$. Then

$$j_{(\mu, \mathbf{m} + m\mathbf{e}^n)}(xe^{i\omega}) \ll_{(\mu, \mathbf{m}), X, \epsilon, n} \left(\frac{2\pi e x^{\frac{1}{n}}}{|m| + 1} \right)^{n|m|} (|m| + 1)^{A+n\epsilon} x^{B+2\epsilon} e^{|\omega|(C+2\epsilon)}$$

for all $x < X$.

4.3. The Hankel transform $\mathcal{H}_{(\mu, \delta)}$ and the Bessel kernel $J_{(\mu, \delta)}$.

4.3.1. *The definition of $\mathcal{H}_{(\mu, \delta)}$.* Consider the ordered set $(\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}, \leq)$ and define $(\mu_{\alpha, \beta}, \delta_{\alpha, \beta}) = (\mu, \delta)_{\alpha, \beta}$, B_{α} , $M_{\alpha, \beta}$ and $N_{\alpha, \beta}$ as in Definition 4.1 corresponding to $(\mu, \delta) \in (\mathbb{C} \times \mathbb{Z}/2\mathbb{Z})^n$. We define the following subspaces of $\mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$,

$$(4.31) \quad \mathcal{S}_{\text{sis}}^{(\mu, \delta), \delta}(\mathbb{R}^\times) = \sum_{\alpha=1}^A \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha, \beta}-1} \text{sgn}(x)^{\delta_{\alpha, \beta}} |x|^{-\mu_{\alpha, \beta}} (\log |x|)^j \mathcal{S}_{\delta_{\alpha, \beta} + \delta}(\mathbb{R}).$$

$$(4.32) \quad \begin{aligned} \mathcal{S}_{\text{sis}}^{(\mu, \delta)}(\mathbb{R}^\times) &= \mathcal{S}_{\text{sis}}^{(\mu, \delta), 0}(\mathbb{R}^\times) \oplus \mathcal{S}_{\text{sis}}^{(\mu, \delta), 1}(\mathbb{R}^\times) \\ &= \sum_{\alpha=1}^A \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha, \beta}-1} \text{sgn}(x)^{\delta_{\alpha, \beta}} |x|^{-\mu_{\alpha, \beta}} (\log |x|)^j \mathcal{S}(\mathbb{R}). \end{aligned}$$

From the definition (4.8) of $\mathcal{T}_{\text{sis}}^\lambda(\mathbb{R}_+)$, together with $\mathcal{S}_\delta(\mathbb{R}) = \text{sgn}(x)^\delta \mathcal{S}_\delta(\overline{\mathbb{R}}_+)$ and $\mathcal{S}_\delta(\overline{\mathbb{R}}_+) = x^\delta \mathcal{S}_0(\overline{\mathbb{R}}_+)$, we have

$$(4.33) \quad \mathcal{S}_{\text{sis}}^{(\mu, \delta), \delta}(\mathbb{R}^\times) = \text{sgn}(x)^\delta \mathcal{T}_{\text{sis}}^{\mu - (\delta + \delta \mathbf{e}^n)}(\mathbb{R}_+).$$

The following theorem gives the definition of the Hankel transform $\mathcal{H}_{(\mu, \delta)}$, which maps $\mathcal{S}_{\text{sis}}^{(-\mu, \delta)}(\mathbb{R}^\times)$ onto $\mathcal{S}_{\text{sis}}^{(\mu, \delta)}(\mathbb{R}^\times)$ bijectively.

THEOREM 4.15. *Let $(\mu, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$. Suppose $v \in \mathcal{S}_{\text{sis}}^{(-\mu, \delta)}(\mathbb{R}^\times)$. Then there exists a unique function $\Upsilon \in \mathcal{S}_{\text{sis}}^{(\mu, \delta)}(\mathbb{R}^\times)$ satisfying the following two identities,*

$$(4.34) \quad \mathcal{M}_\delta \Upsilon(s) = G_{(\mu, \delta + \delta e^n)}(s) \mathcal{M}_\delta v(1-s), \quad \delta \in \mathbb{Z}/2\mathbb{Z}.$$

We call Υ the Hankel transform of v over \mathbb{R}^\times of index (μ, δ) and write $\mathcal{H}_{(\mu, \delta)} v = \Upsilon$. Moreover, we have the Hankel inversion formula

$$(4.35) \quad \mathcal{H}_{(\mu, \delta)} v = \Upsilon, \quad \mathcal{H}_{(-\mu, \delta)} \Upsilon = v.$$

PROOF. Recall that

$$\mathcal{M}_\delta v(s) = 2\mathcal{M}v_\delta(s).$$

In view of (4.33), one has $v_\delta \in \mathcal{S}_{\text{sis}}^{-\mu - (\delta + \delta e^n)}(\mathbb{R}_+)$. Applying Proposition 4.9, there is a unique function $\Upsilon_\delta \in \mathcal{S}_{\text{sis}}^{\mu - (\delta + \delta e^n)}(\mathbb{R}_+)$ satisfying

$$\mathcal{M}\Upsilon_\delta(s) = G_{(\mu, \delta + \delta e^n)}(s) \mathcal{M}v_\delta(1-s).$$

According to (4.33), $\Upsilon(x) = \Upsilon_0(|x|) + \text{sgn}(x) \Upsilon_1(|x|)$ lies in $\mathcal{S}_{\text{sis}}^{(\mu, \delta), 0}(\mathbb{R}^\times) \oplus \mathcal{S}_{\text{sis}}^{(\mu, \delta), 1}(\mathbb{R}^\times) = \mathcal{S}_{\text{sis}}^{(\mu, \delta)}(\mathbb{R}^\times)$. Clearly, Υ satisfies (4.34). Moreover, (4.35) follows immediately from (4.10) in Proposition 4.9. Q.E.D.

COROLLARY 4.16. *Let $(\mu, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$. Suppose that $\varphi \in \mathcal{S}_{\text{sis}}^{-\mu - (\delta + \delta e^n)}(\mathbb{R}_+)$ and $v(x) = \text{sgn}(x)^\delta \varphi(|x|)$. Then*

$$\mathcal{H}_{(\mu, \delta)} v(\pm x) = (\pm)^\delta h_{(\mu, \delta + \delta e^n)} \varphi(x), \quad x \in \mathbb{R}_+.$$

4.3.2. The Bessel kernel $J_{(\mu, \delta)}$. Let $(\mu, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$. We define

$$(4.36) \quad J_{(\mu, \delta)}(\pm x) = \frac{1}{2} \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} (\pm)^\delta j_{(\mu, \delta + \delta e^n)}(x), \quad x \in \mathbb{R}_+,$$

or equivalently,

$$(4.37) \quad J_{(\mu, \delta)}(x) = \frac{1}{2} \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \text{sgn}(x)^\delta j_{(\mu, \delta + \delta e^n)}(|x|), \quad x \in \mathbb{R}^\times.$$

Some properties of $J_{(\mu, \delta)}$ are summarized as below.

PROPOSITION 4.17. *Let $(\mu, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$.*

(1). *Let $(\mu, \delta) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$. We have*

$$J_{(\mu - \mu e^n, \delta - \delta e^n)}(x) = \text{sgn}(x)^\delta |x|^\mu J_{(\mu, \delta)}(x).$$

(2). *$J_{(\mu, \delta)}(x)$ is a real analytic function of x on \mathbb{R}^\times as well as an analytic function of μ on \mathbb{C}^n .*

(3). *Assume that μ satisfies the condition*

$$(4.38) \quad \min \{\Re \mu_l\} + 1 > \max \{\Re \mu_l\}.$$

Then for $v \in \mathcal{S}_{\text{sis}}^{(-\mu, \delta)}(\mathbb{R}^\times)$

$$(4.39) \quad \mathcal{H}_{(\mu, \delta)} v(x) = \int_{\mathbb{R}^\times} v(y) J_{(\mu, \delta)}(xy) dy.$$

Moreover, if $v \in \mathcal{S}(\mathbb{R}^\times)$, then (4.39) remains true for any index $\mu \in \mathbb{C}^n$.

EXAMPLE 4.18. For $n = 1$, we have

$$J_{(0,0)}(x) = e(x).$$

For $n = 2$, (4.15) and (4.36) yield

$$J_{(\mu, -\mu, \delta, 0)}(\pm x) = J(2\pi\sqrt{x}; +, \pm, \mu, -\mu) + (-)^\delta J(2\pi\sqrt{x}; -, \mp, \mu, -\mu)$$

for $x \in \mathbb{R}_+$, $\mu \in \mathbb{C}$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$. In view of Example 4.8, for $x \in \mathbb{R}_+$, we have

$$J_{(\mu, -\mu, \delta, 0)}(x) = \begin{cases} -\frac{\pi}{\sin(\pi\mu)} (J_{2\mu}(4\pi\sqrt{x}) - J_{-2\mu}(4\pi\sqrt{x})), & \text{if } \delta = 0, \\ \frac{\pi i}{\cos(\pi\mu)} (J_{2\mu}(4\pi\sqrt{x}) + J_{-2\mu}(4\pi\sqrt{x})), & \text{if } \delta = 1, \end{cases}$$

where the right hand side is replaced by its limit if $2\mu \in \delta + 2\mathbb{Z}$, and

$$J_{(\mu, -\mu, \delta, 0)}(-x) = \begin{cases} 4 \cos(\pi\mu) K_{2\mu}(4\pi\sqrt{x}), & \text{if } \delta = 0, \\ -4i \sin(\pi\mu) K_{2\mu}(4\pi\sqrt{x}), & \text{if } \delta = 1. \end{cases}$$

Observe that for $m \in \mathbb{N}$

$$J_{(\frac{1}{2}m, -\frac{1}{2}m, \delta(m)+1, 0)}(x) = 2\pi i^{m+1} J_m(4\pi\sqrt{x}), \quad J_{(\frac{1}{2}m, -\frac{1}{2}m, \delta(m)+1, 0)}(-x) = 0.$$

4.4. The Hankel transform $\mathcal{H}_{(\mu, m)}$ and the Bessel kernel $J_{(\mu, m)}$.

4.4.1. *The definition of $\mathcal{H}_{(\mu, m)}$.* Consider now the ordered set $(\mathbb{C} \times \mathbb{Z}, \leq)$ and define $(2\mu_{\alpha\beta}, m_{\alpha\beta}) = (2\mu, m)_{\alpha\beta}$, B_{α} , $M_{\alpha\beta}$ and $N_{\alpha\beta}$ as in Definition 4.1 corresponding to $(2\mu, m) \in (\mathbb{C} \times \mathbb{Z})^n$. We define the following subspace of $\mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$,

$$(4.40) \quad \mathcal{S}_{\text{sis}}^{(\mu, m)}(\mathbb{C}^\times) = \sum_{\alpha=1}^A \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha\beta}-1} [z]^{-m_{\alpha\beta}} \|z\|^{-\mu_{\alpha\beta}} (\log |z|)^j \mathcal{S}(\mathbb{C}).$$

The projection via the m -th Fourier coefficient maps $\mathcal{S}_{\text{sis}}^{(\mu, m)}(\mathbb{C}^\times)$ onto the space

$$(4.41) \quad \mathcal{S}_{\text{sis}}^{(\mu, m), m}(\mathbb{C}^\times) = \sum_{\alpha=1}^A \sum_{\beta=1}^{B_\alpha} \sum_{j=0}^{N_{\alpha\beta}-1} [z]^{-m_{\alpha\beta}} \|z\|^{-\mu_{\alpha\beta}} (\log |z|)^j \mathcal{S}_{m_{\alpha\beta}+m}(\mathbb{C}).$$

From the definition (4.8) of $\mathcal{S}_{\text{sis}}^l(\mathbb{R}_+)$, along with $\mathcal{S}_m(\mathbb{C}) = [z]^m \mathcal{S}_m(\overline{\mathbb{R}}_+)$ and $\mathcal{S}_m(\overline{\mathbb{R}}_+) = x^{|m|} \mathcal{S}_0(\overline{\mathbb{R}}_+)$, we have

$$(4.42) \quad \mathcal{S}_{\text{sis}}^{(\mu, m), m}(\mathbb{C}^\times) = [z]^m \mathcal{S}_{\text{sis}}^{2\mu - \|m + m\epsilon^n\|}(\mathbb{R}_+).$$

The following theorem gives the definition of the Hankel transform $\mathcal{H}_{(\mu, m)}$, which maps $\mathcal{S}_{\text{sis}}^{(-\mu, -m)}(\mathbb{C}^\times)$ onto $\mathcal{S}_{\text{sis}}^{(\mu, m)}(\mathbb{C}^\times)$ bijectively.

THEOREM 4.19. *Let $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$. Suppose $v \in \mathcal{S}_{\text{sis}}^{(-\mu, -m)}(\mathbb{C}^\times)$. Then there exists a unique function $\Upsilon \in \mathcal{S}_{\text{sis}}^{(\mu, m)}(\mathbb{C}^\times)$ satisfying the following sequence of identities,*

$$(4.43) \quad \mathcal{M}_{-m}\Upsilon(2s) = G_{(\mu, m+me^n)}(s)\mathcal{M}_m v(2(1-s)), \quad m \in \mathbb{Z}.$$

We call Υ the Hankel transform of v over \mathbb{C}^\times of index (μ, m) and write $\mathcal{H}_{(\mu, m)}v = \Upsilon$. Moreover, we have the Hankel inversion formula

$$(4.44) \quad \mathcal{H}_{(\mu, m)}v = \Upsilon, \quad \mathcal{H}_{(-\mu, -m)}\Upsilon = v.$$

PROOF. Recall that

$$\mathcal{M}_m v(s) = 4\pi \mathcal{M}_{v-m}(s).$$

In view of (4.42), we have $v_{-m} \in \mathcal{S}_{\text{sis}}^{-2\mu - \|m+me^n\|}(\mathbb{R}_+)$. Applying Proposition 4.10, one finds that there is a unique function $\Upsilon_m \in \mathcal{S}_{\text{sis}}^{2\mu - \|m+me^n\|}(\mathbb{R}_+)$ satisfying

$$\mathcal{M}\Upsilon_m(2s) = G_{(\mu, m+me^n)}(s)\mathcal{M}_{v-m}(2(1-s)).$$

According to Lemma 3.8, to show that the Fourier series $\Upsilon(xe^{i\phi}) = \sum_{m \in \mathbb{Z}} \Upsilon_m(x)e^{im\phi}$ lies in $\mathcal{S}_{\text{sis}}^{(\mu, m)}(\mathbb{C}^\times)$, it suffices to verify that $G_{(\mu, m+me^n)}(s)\mathcal{M}_{v-m}(2(1-s))$ rapidly decay with respect to m , uniformly on vertical strips. This follows from the uniform rapid decay of $\mathcal{M}_{v-m}(2(1-s))$ along with the uniform moderate growth of $G_{(\mu, m+me^n)}(s)$ ((2.16) in Lemma 2.2) in the m aspect on vertical strips.

Finally, (4.12) in Proposition 4.10 implies (4.44). Q.E.D.

COROLLARY 4.20. *Let $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$ and $m \in \mathbb{Z}$. Suppose $\varphi \in \mathcal{S}_{\text{sis}}^{-2\mu - \|m+me^n\|}(\mathbb{R}_+)$ and $v(z) = [z]^{-m}\varphi(|z|)$. Then*

$$\mathcal{H}_{(\mu, m)}v(xe^{i\phi}) = e^{im\phi} f_{(\mu, m+me^n)}\varphi(x), \quad x \in \mathbb{R}_+, \phi \in \mathbb{R}/2\pi\mathbb{Z}.$$

4.4.2. The Bessel kernel $J_{(\mu, m)}$. For $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$, we define

$$(4.45) \quad J_{(\mu, m)}(xe^{i\phi}) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} j_{(\mu, m+me^n)}(x)e^{im\phi},$$

or equivalently,

$$(4.46) \quad J_{(\mu, m)}(z) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} j_{(\mu, m+me^n)}(|z|)[z]^m.$$

Lemma 4.13 secures the absolute convergence of this series.

PROPOSITION 4.21. *Let $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$.*

(1). *Let $(\mu, m) \in \mathbb{C} \times \mathbb{Z}$. We have*

$$J_{(\mu - \mu e^n, m - me^n)}(z) = [z]^m \|z\|^\mu J_{(\mu, m)}(z).$$

(2). *$J_{(\mu, m)}(z)$ is a real analytic function of z on \mathbb{C}^\times as well as an analytic function of μ on \mathbb{C}^n .*

(3). *Assume that μ satisfies the following condition*

$$(4.47) \quad \min \{\Re \mu_l\} + 1 > \max \{\Re \mu_l\}.$$

Suppose $v \in \mathcal{S}_{\text{sis}}^{(-\mu, -m)}(\mathbb{C}^\times)$. Then

$$(4.48) \quad \Upsilon(xe^{i\phi}) = \int_0^\infty \int_0^{2\pi} v(ye^{i\theta}) J_{(\mu, m)}(xe^{i\phi} ye^{i\theta}) \cdot 2y d\theta dy,$$

or equivalently,

$$(4.49) \quad \Upsilon(z) = \int_{\mathbb{C}^\times} v(u) J_{(\mu, m)}(zu) du.$$

Moreover, (4.48) and (4.49) still hold true for any index $\mu \in \mathbb{C}$ if $v \in \mathcal{S}(\mathbb{C}^\times)$.

PROOF. (1). This is clear.

(2). In (4.45), with abuse of notation, we view x and ϕ as complex variables on \mathbb{U} and $\mathbb{C}/2\pi\mathbb{Z}$ respectively, $j_{(\mu, m+me^n)}(x)$ and $e^{im\phi}$ as analytic functions. Then Lemma 4.14 implies that the series in (4.45) is absolutely convergent, compactly with respect to both x and ϕ , and therefore $J_{(\mu, m)}(xe^{i\phi})$ is an analytic function of x and ϕ . In particular, $J_{(\mu, m)}(z)$ is a *real analytic* function of z on \mathbb{C}^\times .

Moreover, in Lemma 4.13, one may allow μ to vary in an ϵ -ball in \mathbb{C}^n and choose the implied constant in the estimate to be uniformly bounded with respect to μ . This implies that the series in (4.45) is convergent compactly in the μ aspect. Therefore, $J_{(\mu, m)}(z)$ is an analytic function of μ on \mathbb{C}^n .

(3). It follows from (4.42) that $v_{-m} \in \mathcal{S}_{\text{sis}}^{-2\mu - \|m+me^n\|}(\mathbb{R}_+)$. Moreover, one observes that $(\mu, m + me^n)$ satisfies the condition (4.27) due to (4.47). Therefore, in conjunction with Proposition 4.10, (4.28) implies

$$\Upsilon_m(x) = 2 \int_0^\infty v_{-m}(y) j_{(\mu, m+me^n)}(xy) y dy.$$

Hence

$$\begin{aligned} \Upsilon(xe^{i\phi}) &= \sum_{m \in \mathbb{Z}} \Upsilon_m(x) e^{im\phi} \\ &= \sum_{m \in \mathbb{Z}} \frac{1}{\pi} \int_0^\infty \int_0^{2\pi} v(ye^{i\theta}) j_{(\mu, m+me^n)}(xy) e^{im(\phi+\theta)} y d\theta dy. \end{aligned}$$

The estimate of $j_{(\mu, m+me^n)}$ in Lemma 4.13 implies that the above series of integrals converges absolutely. On interchanging the order of summation and integration, one obtains (4.48) in view of the definition (4.45) of $J_{(\mu, m)}$.

Note that in the case $v \in \mathcal{S}(\mathbb{C}^\times)$, one has $v_{-m} \in \mathcal{S}(\mathbb{R}_+)$, and therefore (4.28) can be applied unconditionally. Q.E.D.

EXAMPLE 4.22. Let $n = 1$. From (4.29), we have

$$j_{(0, m)}(x) = \begin{cases} (-)^d 2\pi J_{2d}(4\pi x), & \text{if } |m| = 2d, \\ (-)^d 2\pi i J_{2d+1}(4\pi x), & \text{if } |m| = 2d + 1. \end{cases}$$

The following expansions ([Wat, 2.22 (3, 4)])

$$\cos(x \cos \phi) = J_0(x) + 2 \sum_{d=1}^{\infty} (-)^d J_{2d}(x) \cos(2d\phi),$$

$$\sin(x \cos \phi) = 2 \sum_{d=0}^{\infty} (-1)^d J_{2d+1}(x) \cos((2d+1)\phi),$$

imply

$$J_{(0,0)}(xe^{i\phi}) = \cos(4\pi x \cos \phi) + i \sin(4\pi x \cos \phi) = e(2x \cos \phi),$$

or equivalently,

$$J_{(0,0)}(z) = e(z + \bar{z}).$$

We remark that the two expansions [Wat, 2.22 (3, 4)] can be incorporated into

$$e^{ix \cos \phi} = \sum_{m=-\infty}^{\infty} i^m J_m(x) e^{im\phi}.$$

4.5. Concluding remarks.

4.5.1. *Connection formulae.* From the various connection formulae (4.15, 4.24, 4.45, 4.46) that we have derived so far, one can connect the Bessel kernel $J_{(\mu,m)}(z)$ to the Bessel functions $J(x; \mathfrak{s}, \lambda)$ of doubled rank $2n$. However, in contrast to the expression of $J_{(\mu,\delta)}(\pm x)$ by a *finite* sum of $J(2\pi x^{\frac{1}{n}}; \mathfrak{s}, \mu)$ (see (4.15, 4.36, 4.37)), which enables us to reduce the study of $J_{(\mu,\delta)}(x)$ to that of $J(x; \mathfrak{s}, \lambda)$ given in [Qi1], these connection formulae yield an expression of $J_{(\mu,m)}(xe^{i\phi})$ in terms of an *infinite* series involving the Bessel functions $J(2\pi x^{\frac{1}{n}}; \mathfrak{s}, \mu)$ of rank $2n$, so a similar reduction for $J_{(\mu,m)}(z)$ does not exist from this approach.

In §7, we shall prove two alternative connection formulae that relate $J_{(\mu,m)}(z)$ to the two kinds of Bessel functions of rank n and positive sign. These kinds of Bessel functions arise in [Qi1, §7] as solutions of the Bessel equation of positive sign.

4.5.2. *Asymptotics of Bessel kernels.* Using the connection formulae between the Bessel kernel $J_{(\mu,\delta)}(x)$ and Bessel functions $J(x; \mathfrak{s}, \lambda)$ along with the asymptotics of the latter, the asymptotic of $J_{(\mu,\delta)}(x)$ is readily established in [Qi1, Theorem 5.13, 9.4]. With the help of the second connection formula for $J_{(\mu,m)}(z)$ in §7.2, we shall present in §8 the asymptotic of $J_{(\mu,m)}(z)$ as an application of the asymptotic expansions of Bessel functions of the second kind [Qi1, Theorem 7.27].

4.5.3. *Normalizations of indices.* Usually, it is convenient to normalize the indices in $J(x; \mathfrak{s}, \lambda)$, $j_{(\mu,\delta)}(x)$, $j_{(\mu,m)}(x)$, $J_{(\mu,\delta)}(x)$ and $J_{(\mu,m)}(z)$ so that $\lambda, \mu \in \mathbb{L}^{n-1}$. Furthermore, without loss of generality, the assumptions $\delta_n = 0$ and $m_n = 0$ may also be imposed for $J_{(\mu,\delta)}(x)$ and $J_{(\mu,m)}(z)$ respectively. These normalizations are justified by Lemma 4.6, (4.14), (4.23), Lemma 4.17 (1) and 4.21 (1).

5. Fourier type integral transforms

In this section, we shall introduce an alternative perspective of Hankel transforms. We shall first show how to construct Hankel transforms from the Fourier transform and Miller-Schmid transforms. From this, we shall express the Hankel transforms $\mathcal{H}_{(\mu,\delta)}$ and $\mathcal{H}_{(\mu,m)}$ in terms of certain Fourier type integral transforms, assuming that the components of $\Re \mu$ are *strictly* decreasing.

5.1. The Fourier transform and rank-one Hankel transforms. For either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, we have seen in Example 4.18 and 4.22 that $J_{(0,0)}$ is exactly the inverse Fourier kernel, namely

$$J_{(0,0)}(x) = e(\Lambda(x)), \quad x \in \mathbb{F},$$

with $\Lambda(x)$ defined by (2.37). Therefore, in view of Proposition 4.17 (3) and 4.21 (3), $\mathcal{H}_{(0,0)}$ is precisely the inverse Fourier transform over the Schwartz space $\mathcal{S}_{\text{sis}}^{(0,0)}(\mathbb{F}^\times) = \mathcal{S}(\mathbb{F})$. The following lemma is a consequence of Theorem 4.15 and 4.19.

LEMMA 5.1. *Let $v \in \mathcal{S}(\mathbb{F})$. If $\mathbb{F} = \mathbb{R}$, then the Fourier transform \hat{v} of v can be determined by the following two identities*

$$\mathcal{M}_\delta \hat{v}(s) = (-)^\delta G_\delta(s) \mathcal{M}_\delta v(1-s), \quad \delta \in \mathbb{Z}/2\mathbb{Z}.$$

If $\mathbb{F} = \mathbb{C}$, then the Fourier transform \hat{v} of v can be determined by the following sequence of identities

$$\mathcal{M}_{-m} \hat{v}(2s) = (-)^m G_m(s) \mathcal{M}_m v(2(1-s)), \quad m \in \mathbb{Z}.$$

It is convenient for our purpose to introduce the renormalized rank-one Hankel transforms $\mathcal{S}_{(\mu,\epsilon)}$ and $\mathcal{S}_{(\mu,k)}$ as follows.

LEMMA 5.2. *Let $(\mu, \epsilon) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$ and $(\mu, k) \in \mathbb{C} \times \mathbb{Z}$.*

(1). For $v \in \text{sgn}(x)^\epsilon |x|^\mu \mathcal{S}(\mathbb{R})$, define $\mathcal{S}_{(\mu,\epsilon)} v(x) = |x|^\mu \mathcal{H}_{(\mu,\epsilon)} v(x)$. Then

$$(5.1) \quad \mathcal{M}_\delta \mathcal{S}_{(\mu,\epsilon)} v(s) = G_{\epsilon+\delta}(s) \mathcal{M}_\delta v(1-s-\mu), \quad \delta \in \mathbb{Z}/2\mathbb{Z},$$

and $\mathcal{S}_{(\mu,\epsilon)}$ sends $\text{sgn}(x)^\epsilon |x|^\mu \mathcal{S}(\mathbb{R})$ onto $\text{sgn}(x)^\epsilon \mathcal{S}(\mathbb{R})$ bijectively. Furthermore,

$$\mathcal{S}_{(\mu,\epsilon)} v(x) = \text{sgn}(x)^\epsilon \int_{\mathbb{R}^\times} \text{sgn}(y)^\epsilon |y|^{-\mu} v(y) e(xy) dy = \text{sgn}(x)^\epsilon \mathcal{F}(\text{sgn}^\epsilon | \cdot |^{-\mu} v)(-x).$$

(2). For $v \in [z]^k \|z\|^\mu \mathcal{S}(\mathbb{C})$, define $\mathcal{S}_{(\mu,k)} v(z) = \|z\|^\mu \mathcal{H}_{(\mu,k)} v(z)$. Then

$$(5.2) \quad \mathcal{M}_{-m} \mathcal{S}_{(\mu,k)} v(2s) = G_{k+m}(s) \mathcal{M}_m v(2(1-s-\mu)), \quad m \in \mathbb{Z}.$$

and $\mathcal{S}_{(\mu,k)}$ sends $[z]^k \|z\|^\mu \mathcal{S}(\mathbb{C})$ onto $[z]^{-k} \mathcal{S}(\mathbb{C})$ bijectively. Furthermore,

$$\mathcal{S}_{(\mu,k)} v(z) = [z]^{-k} \int_{\mathbb{C}^\times} [u]^{-k} \|u\|^{-\mu} v(u) e(zu + \overline{zu}) du = [z]^{-k} \mathcal{F}([\cdot]^{-k} \| \cdot \|^{-\mu} v)(-z).$$

LEMMA 5.3. *Let $(\mu, \epsilon) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$ and $(\mu, k) \in \mathbb{C} \times \mathbb{Z}$.*

(1). Let $\delta \in \mathbb{Z}/2\mathbb{Z}$. Suppose that $\varphi \in x^\mu \mathcal{S}_{\delta+\epsilon}(\overline{\mathbb{R}}_+)$ and $v(x) = \text{sgn}(x)^\delta \varphi(|x|)$. Then

$$\begin{aligned} \mathcal{S}_{(\mu,\epsilon)} v(\pm x) &= (\pm)^\delta \int_{\mathbb{R}_+} y^{-\mu} \varphi(y) j_{(0,\delta+\epsilon)}(xy) dy \\ &= \begin{cases} (\pm)^\epsilon 2 \int_{\mathbb{R}_+} y^{-\mu} \varphi(y) \cos(xy) dy, & \text{if } \delta = \epsilon, \\ (\pm)^{\epsilon+1} 2i \int_{\mathbb{R}_+} y^{-\mu} \varphi(y) \sin(xy) dy, & \text{if } \delta = \epsilon + 1. \end{cases} \end{aligned}$$

The transform $\mathcal{S}_{(\mu,\epsilon)}$ is a bijective map from $\text{sgn}(x)^\epsilon |x|^\mu \mathcal{S}_\delta(\mathbb{R})$ onto $\text{sgn}(x)^\epsilon \mathcal{S}_\delta(\mathbb{R})$.

(2). Let $m \in \mathbb{Z}$. Suppose that $\varphi \in x^{2\mu} \mathcal{S}_{-m-k}(\overline{\mathbb{R}}_+)$ and $v(z) = [z]^{-m} \varphi(|z|)$. Then

$$\begin{aligned} \mathcal{S}_{(\mu,k)} v(xe^{i\phi}) &= 2e^{im\phi} \int_{\mathbb{R}_+} y^{1-2\mu} \varphi(y) j_{(0,m+k)}(xy) dy \\ &= 4\pi i^{m+k} e^{im\phi} \int_{\mathbb{R}_+} y^{1-2\mu} \varphi(y) J_{m+k}(4\pi xy) dy. \end{aligned}$$

The transform $\mathcal{S}_{(\mu,k)}$ is a bijective map from $[z]^k \|z\|^\mu \mathcal{S}_m(\mathbb{C})$ onto $[z]^{-k} \mathcal{S}_{-m}(\mathbb{C})$.

5.2. Miller-Schmid transforms. In [MS1, §6], certain transforms over \mathbb{R} , which play an important role in the proof of the Voronoï summation formula in their subsequent work [MS3, MS4], are introduced by Miller and Schmid. Here, we shall first recollect their construction of these transforms with slight modifications, and then define similar transforms over \mathbb{C} in a parallel way.

5.2.1. *The Miller-Schmid transform $\mathcal{T}_{(\mu,\epsilon)}$.*

LEMMA 5.4. Let $(\mu, \epsilon) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$.

(1). For any $v \in \mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$ there is a unique function $\Upsilon \in \mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$ satisfying the following two identities,

$$(5.3) \quad \mathcal{M}_\delta \Upsilon(s) = G_{\epsilon+\delta}(s) \mathcal{M}_\delta v(s+\mu), \quad \delta \in \mathbb{Z}/2\mathbb{Z}.$$

We write $\Upsilon = \mathcal{T}_{(\mu,\epsilon)} v$ and call $\mathcal{T}_{(\mu,\epsilon)}$ the Miller-Schmid transform over \mathbb{R} of index (μ, ϵ) .

(2). Let $\lambda \in \mathbb{C}$. Suppose $v \in \text{sgn}(x)^\delta |x|^{-\lambda} (\log |x|)^j \mathcal{S}(\mathbb{R})$. If $\Re \lambda < \Re \mu - \frac{1}{2}$, then

$$(5.4) \quad \begin{aligned} \mathcal{T}_{(\mu,\epsilon)} v(x) &= \text{sgn}(x)^\epsilon \int_{\mathbb{R}^\times} \text{sgn}(y)^\epsilon |y|^{-\mu} v(y^{-1}) e(xy) d^\times y \\ &= \text{sgn}(x)^\epsilon \mathcal{F}(\text{sgn}^\epsilon | \cdot |^{-\mu-1} v^\#)(-x), \end{aligned}$$

with the notation $v^\#(x) = v(x^{-1})$.

(3). Suppose that $\Re \lambda < \Re \mu$. Then the integral in (5.4) is absolutely convergent and (5.4) remains valid for any $v \in \text{sgn}(x)^\delta |x|^{-\lambda} (\log |x|)^j \mathcal{S}(\mathbb{R})$.

(4). Suppose that $\Re \mu > 0$. Define the function space

$$\mathcal{T}_{\text{sis}}(\mathbb{R}^\times) = \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \sum_{\Re \lambda \leq 0} \sum_{j \in \mathbb{N}} \text{sgn}(x)^\delta |x|^{-\lambda} (\log |x|)^j \mathcal{S}(\mathbb{R}).$$

Then the transform $\mathcal{T}_{(\mu,\epsilon)}$ sends $\mathcal{T}_{\text{sis}}(\mathbb{R}^\times)$ into itself. Moreover, (5.4) also holds true for any $v \in \mathcal{T}_{\text{sis}}(\mathbb{R}^\times)$, wherein the integral absolutely converges.

PROOF. Following the ideas in the proofs of Proposition 4.9 and Theorem 4.15, one may prove (1). Actually, the case here is much easier!

As for (2), we have

$$\begin{aligned} \mathcal{T}_{(\mu,\epsilon)} v(x) &= \frac{1}{4\pi i} \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \int_{\mathbb{R}^\times} v(y) |y|^\mu \cdot \text{sgn}(xy)^\delta \int_{\mathcal{C}_{(0,\delta+\epsilon)}} G_{\delta+\epsilon}(s) |y|^s |x|^{-s} ds d^\times y \\ &= \int_{\mathbb{R}^\times} v(y) |y|^\mu J_{(0,\epsilon)}(xy^{-1}) d^\times y, \end{aligned}$$

provided that the double integral is absolutely convergent. In order to guarantee the convergence of the integral over $d^\times y$, the integral contour $\mathcal{C}_{(0,\delta+\epsilon)}$ is required to lie in the right half-plane $\{s : \Re s > \Re(\lambda - \mu)\}$. In view of Definition 4.2, such a choice of $\mathcal{C}_{(0,\delta+\epsilon)}$ is permissible since $\Re(\lambda - \mu) < -\frac{1}{2}$ according to our assumption. Finally, the change of variables from y to y^{-1} , along with the formula $J_{(0,\epsilon)}(x) = \operatorname{sgn}(x)^\epsilon e(x)$, yields (5.4).

For the case $\Re \lambda < \Re \mu$ in (3), the absolute convergence of the integral in (5.4) is obvious. The validity of (5.4) follows from the analyticity with respect to μ .

Observe that, under the isomorphism established by $\mathcal{M}^\mathbb{R}$ in Lemma 3.6, $\mathcal{T}_{\text{sis}}(\mathbb{R}^\times)$ corresponds to the subspace of $\mathcal{M}_{\text{sis}}^\mathbb{R}$ consisting of pairs of meromorphic functions (H_0, H_1) such that the poles of both H_0 and H_1 lie in the left half-plane $\{s : \Re s \leq 0\}$ (see Lemma 3.7). Then the first assertion in (4) is clear, since the map that corresponds to $\mathcal{T}_{(\mu,\epsilon)}$ is given by $(H_0(s), H_1(s)) \mapsto (G_\epsilon(s)H_0(s+\mu), G_{\epsilon+1}(s)H_1(s+\mu))$ and sends the subspace of $\mathcal{M}_{\text{sis}}^\mathbb{R}$ described above into itself. The second assertion in (4) immediately from (3). Q.E.D.

Similar to Lemma 5.3 (1), we have the following lemma.

LEMMA 5.5. *Let $(\mu, \epsilon) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$ such that $\Re \mu > 0$. For $\delta \in \mathbb{Z}/2\mathbb{Z}$ define $\mathcal{T}_{\text{sis}}^\delta(\mathbb{R}^\times)$ to be the space of functions in $\mathcal{T}_{\text{sis}}(\mathbb{R}^\times)$ satisfying the condition (2.17). For $v \in \mathcal{T}_{\text{sis}}^\delta(\mathbb{R}^\times)$, we write $v(x) = \operatorname{sgn}(x)^\delta \varphi(|x|)$. Then*

$$\begin{aligned} \mathcal{T}_{(\mu,\epsilon)} v(\pm x) &= (\pm)^\delta \int_{\mathbb{R}_+} y^{-\mu} \varphi(y^{-1}) j_{(0,\delta+\epsilon)}(xy) d^\times y \\ &= \begin{cases} (\pm)^\delta 2 \int_{\mathbb{R}_+} y^{-\mu} \varphi(y^{-1}) \cos(xy) d^\times y, & \text{if } \delta = \epsilon, \\ (\pm)^\delta 2i \int_{\mathbb{R}_+} y^{-\mu} \varphi(y^{-1}) \sin(xy) d^\times y, & \text{if } \delta = \epsilon + 1. \end{cases} \end{aligned}$$

The transform $\mathcal{T}_{(\mu,\epsilon)}$ sends $\mathcal{T}_{\text{sis}}^\delta(\mathbb{R}^\times)$ into itself.

5.2.2. The Miller-Schmid transform $\mathcal{T}_{(\mu,k)}$. In parallel to Lemma 5.4, the following lemma defines the Miller-Schmid transform $\mathcal{T}_{(\mu,k)}$ over \mathbb{C} and gives its connection to the Fourier transform over \mathbb{C} .

LEMMA 5.6. *Let $(\mu, k) \in \mathbb{C} \times \mathbb{Z}$.*

(1). *For any $v \in \mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$ there is a unique function $\Upsilon \in \mathcal{S}_{\text{sis}}(\mathbb{C}^\times)$ satisfying the following sequence of identities,*

$$(5.5) \quad \mathcal{M}_{-m} \Upsilon(2s) = G_{m+k}(s) \mathcal{M}_{-m} v(2(s+\mu)), \quad m \in \mathbb{Z}.$$

We write $\Upsilon = \mathcal{T}_{(\mu,k)} v$ and call $\mathcal{T}_{(\mu,k)}$ the Miller-Schmid transform over \mathbb{C} of index (μ, k) .

(2). *Let $\lambda \in \mathbb{C}$. If $\Re \lambda < 2 \Re \mu$, then*

$$(5.6) \quad \begin{aligned} \mathcal{T}_{(\mu,k)} v(x) &= [z]^k \int_{\mathbb{C}^\times} [u]^k \|u\|^{-\mu} v(u^{-1}) e(zu + \overline{z}u) d^\times u \\ &= [z]^k \mathcal{F}([]^k \| \cdot \|^{-\mu-1} v^\sharp)(-z), \end{aligned}$$

for any $v \in [z]^m |z|^{-\lambda} (\log |z|)^j \mathcal{S}(\mathbb{C})$. Here $v^\sharp(z) = v(z^{-1})$.

(3). When $\Re \lambda < 2\Re \mu$, the integral in (5.6) is absolutely convergent for any $v \in [z]^m |z|^{-\lambda} (\log |z|)^j \mathcal{S}(\mathbb{C})$.

(4). Suppose that $\Re \mu > 0$. Define the function space

$$\mathcal{T}_{\text{sis}}(\mathbb{C}^\times) = \sum_{m \in \mathbb{Z}} \sum_{\Re \lambda \leq 0} \sum_{j \in \mathbb{N}} [z]^m |z|^{-\lambda} (\log |z|)^j \mathcal{S}(\mathbb{C}).$$

Then the transform $\mathcal{T}_{(\mu,k)}$ sends $\mathcal{T}_{\text{sis}}(\mathbb{C}^\times)$ into itself. Moreover, (5.4) also holds true for any $v \in \mathcal{T}_{\text{sis}}(\mathbb{C}^\times)$, wherein the integral absolutely converges.

PROOF. Following literally the same ideas in the proof of Lemma 5.4, one may show this lemma without any difficulty. We only remark that, via the isomorphism $\mathcal{M}^\mathbb{C}$ in Lemma 3.8, $\mathcal{T}_{\text{sis}}(\mathbb{C}^\times)$ corresponds to the subspace of $\mathcal{M}_{\text{sis}}^\mathbb{C}$ consisting of sequences $\{H_m\}_{m \in \mathbb{Z}}$ such that the poles of each H_m lie in the left half-plane $\{s : \Re s \leq \min\{M - |m|, 0\}\}$ for some $M \in \mathbb{N}$ (see Lemma 3.9). Q.E.D.

LEMMA 5.7. Let $(\mu, k) \in \mathbb{C} \times \mathbb{Z}$ be such that $\Re \mu > 0$. For $m \in \mathbb{Z}$ define $\mathcal{T}_{\text{sis}}^m(\mathbb{C}^\times)$ to be the space of functions in $\mathcal{T}_{\text{sis}}(\mathbb{C}^\times)$ satisfying the condition (2.21). For $v \in \mathcal{T}_{\text{sis}}^m(\mathbb{C}^\times)$, we write $v(z) = [z]^m \varphi(|z|)$. Then

$$\begin{aligned} \mathcal{T}_{(\mu,k)} v(xe^{i\phi}) &= 2e^{im\phi} \int_{\mathbb{R}_+} y^{-2\mu} \varphi(y^{-1}) j_{(0,m+k)}(xy) d^\times y \\ &= 4\pi i^{m+k} e^{im\phi} \int_{\mathbb{R}_+} y^{-2\mu} \varphi(y^{-1}) J_{m+k}(4\pi xy) d^\times y. \end{aligned}$$

The transform $\mathcal{T}_{(\mu,k)}$ sends $\mathcal{T}_{\text{sis}}^m(\mathbb{C}^\times)$ into itself.

5.3. Fourier type integral transforms. In the following, we shall derive the Fourier type integral transform expressions for $\mathcal{H}_{(\mu,\delta)}$ and $\mathcal{H}_{(\mu,m)}$ from the Fourier transform (more precisely, the renormalized rank-one Hankel transforms) and the Miller-Schmid transforms.

5.3.1. *The Fourier type transform expression for $\mathcal{H}_{(\mu,\delta)}$.* Let $(\mu, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$. Following [MS3, (6.51)], for $v \in \text{sgn}(x)^{\delta_n} |x|^{\mu_n} \mathcal{S}(\mathbb{R})$, we consider

$$(5.7) \quad \Upsilon(x) = |x|^{-\mu_1} \mathcal{T}_{(\mu_1 - \mu_2, \delta_1)} \circ \dots \circ \mathcal{T}_{(\mu_{n-1} - \mu_n, \delta_{n-1})} \circ \mathcal{S}_{(\mu_n, \delta_n)} v(x).$$

According to Lemma 5.2 (1) and Lemma 5.4 (1), $\mathcal{S}_{(\mu_n, \delta_n)} v$ lies in the space $\text{sgn}(x)^{\delta_n} \mathcal{S}(\mathbb{R})$, whereas each Miller-Schmid transform sends $\mathcal{S}_{\text{sis}}(\mathbb{R}^\times)$ into itself. Thus, one can apply the Mellin transform \mathcal{M}_δ to both sides of (5.7). Using (5.1) and (5.3), some calculations show that the application of \mathcal{M}_δ turns (5.7) exactly into the identities in (4.34) that defines $\mathcal{H}_{(\mu,\delta)}$. Therefore, $\Upsilon = \mathcal{H}_{(\mu,\delta)} v$.

THEOREM 5.8. [MS4, (1.3)]. Let $(\mu, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ be such that $\Re \mu_1 > \dots > \Re \mu_{n-1} > \Re \mu_n$. Suppose $v \in \text{sgn}(x)^{\delta_n} |x|^{\mu_n} \mathcal{S}(\mathbb{R})$. Then

$$(5.8) \quad \mathcal{H}_{(\mu,\delta)} v(x) = \frac{1}{|x|} \int_{\mathbb{R}^{\times n}} v\left(\frac{x_1 \dots x_n}{x}\right) \left(\prod_{l=1}^n \text{sgn}(x_l)^{\delta_l} |x_l|^{-\mu_l} e(x_l) \right) dx_n \dots dx_1,$$

where the integral converges when performed as iterated integral in the indicated order $dx_n dx_{n-1} \dots dx_1$, starting from dx_n , then dx_{n-1} , ..., and finally dx_1 .

PROOF. We first observe that $S_{(\mu_n, \delta_n)} v \in \text{sgn}(x)^{\delta_n} \mathcal{S}(\mathbb{R}) \subset \mathcal{T}_{\text{sis}}(\mathbb{R}^\times)$. For each $l = 1, \dots, n-1$, since $\Re(\mu_l - \mu_{l+1}) > 0$, Lemma 5.4 (4) implies that the transform $\mathcal{T}_{(\mu_l - \mu_{l+1}, \delta_l)}$ sends the space $\mathcal{T}_{\text{sis}}(\mathbb{R}^\times)$ into itself. According to Lemma 5.2 (1) and Lemma 5.4 (3), $S_{(\mu_n, \delta_n)}$ and all the $\mathcal{T}_{(\mu_l - \mu_{l+1}, \delta_l)}$ in (5.7) may be expressed as integral transforms, which are absolutely convergent. From these, the right hand side of (5.7) turns into the integral,

$$\int_{\mathbb{R}^{\times n}} \text{sgn}(x)^{\delta_1} |x|^{-\mu_1} e(xy_1) \left(\prod_{l=1}^{n-1} \text{sgn}(y_l)^{\delta_{l+1} + \delta_l} |y_l|^{\mu_{l+1} - \mu_l - 1} e(y_l^{-1} y_{l+1}) \right) \text{sgn}(y_n)^{\delta_n} |y_n|^{-\mu_n} v(y_n) dy_n \dots dy_1,$$

which converges as iterated integral. Our proof is completed upon making the change of variables $x_1 = xy_1$, $x_{l+1} = y_l^{-1} y_{l+1}$, $l = 1, \dots, n-1$. Q.E.D.

We have the following corollary to Theorem 5.8, which can also be seen from Lemma 5.3 (1) and Lemma 5.5.

COROLLARY 5.9. Let $(\mu, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$. Assume that $\Re \mu_1 > \dots > \Re \mu_{n-1} > \Re \mu_n$. Let $\varphi \in x^{2\mu_n} \mathcal{S}_{\delta + \delta_n}(\overline{\mathbb{R}}_+)$ and $v(x) = \text{sgn}(x)^\delta \varphi(|x|)$. Then

$$(5.9) \quad \mathcal{H}_{(\mu, m)} v(\pm x) = \frac{(\pm)^\delta}{x} \int_{\mathbb{R}_+^n} \varphi\left(\frac{x_1 \dots x_n}{x}\right) \left(\prod_{l=1}^n x_l^{-\mu_l} j_{(0, \delta_l + \delta)}(x_l) \right) dx_n \dots dx_1,$$

with $x \in \mathbb{R}_+$. Here the iterated integration is performed in the indicated order.

5.3.2. The Fourier type transform expression for $\mathcal{H}_{(\mu, m)}$. Let $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$. Using Lemma 5.2 (2) and Lemma 5.6 (1), especially (5.2) and (5.5), one may show that

$$(5.10) \quad \mathcal{H}_{(\mu, m)} v(z) = \|z\|^{-\mu_1} \mathcal{T}_{(\mu_1 - \mu_2, m_1)} \circ \dots \circ \mathcal{T}_{(\mu_{n-1} - \mu_n, m_{n-1})} \circ S_{(\mu_n, m_n)} v(z).$$

THEOREM 5.10. Let $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$ be such that $\Re \mu_1 > \dots > \Re \mu_{n-1} > \Re \mu_n$. Suppose $v \in [z]^{m_n} \|z\|^{\mu_n} \mathcal{S}(\mathbb{C})$. Then

$$(5.11) \quad \mathcal{H}_{(\mu, m)} v(z) = \frac{1}{\|z\|} \int_{\mathbb{C}^{\times n}} \left(\prod_{l=1}^n [z_l]^{-m_l} \|z_l\|^{-\mu_l} e(z_l + \overline{z_l}) \right) v\left(\frac{z_1 \dots z_n}{z}\right) dz_n \dots dz_1,$$

where the integral converges when performed as iterated integral in the indicated order.

PROOF. One applies the same arguments in the proof of Theorem 5.8 using Lemma 5.2 (2) and Lemma 5.6 (3, 4). Q.E.D.

Lemma 5.3 (2) and Lemma 5.7 yield the following corollary.

COROLLARY 5.11. Let $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$ and $m \in \mathbb{Z}$. Assume that $\Re \mu_1 > \dots > \Re \mu_{n-1} > \Re \mu_n$. Let $\varphi \in x^{2\mu_n} \mathcal{S}_{-m - m_n}(\overline{\mathbb{R}}_+)$ and $v(z) = [z]^{-m} \varphi(|z|)$. Then

$$(5.12) \quad \mathcal{H}_{(\mu, m)} v(xe^{i\phi}) = 2^n \frac{e^{im\phi}}{x^2} \int_{\mathbb{R}_+^n} \varphi\left(\frac{x_1 \dots x_n}{x}\right) \left(\prod_{l=1}^n x_l^{-2\mu_l + 1} j_{(0, m_l + m)}(x_l) \right) dx_n \dots dx_1,$$

with $x \in \mathbb{R}_+$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. Here the iterated integration is performed in the indicated order.

6. Integral representations of Bessel kernels

In the previous article [Qi1], when $n \geq 2$, the formal integral representation of the Bessel function $J(x; \boldsymbol{\varsigma}, \boldsymbol{\mu})$ is obtained in symbolic manner from the Fourier type integral in Theorem 5.8, where the assumption $\Re \mu_1 > \dots > \Re \mu_n$ is simply ignored. It is however more straightforward to derive the formal integral representation of the Bessel kernel $J_{(\mu, \delta)}(x)$ from Theorem 5.8. This should be well understood, since $J_{(\mu, \delta)}(x)$ is a finite combination of $J(2\pi|x|^{\frac{1}{n}}; \boldsymbol{\varsigma}, \boldsymbol{\mu})$.

Similarly, Theorem 5.10 also yields a formal integral representation of $J_{(\mu, m)}(z)$. It turns out that one can naturally transform this formal integral into an integral that is absolutely convergent, given that the index $\boldsymbol{\mu}$ satisfies certain conditions. The main reason for the absolute convergence is that $j_{(0, m)}(x) = 2\pi i^m J_m(4\pi x)$ (see (4.29)) decays proportionally to $\frac{1}{\sqrt{x}}$ at infinity (in comparison, $j_{(0, \delta)}(x)$ is equal to either $2 \cos(2\pi x)$ or $2i \sin(2\pi x)$).

Assumptions and notations. Let $n \geq 2$. Assume that $\boldsymbol{\mu} \in \mathbb{L}^{n-1}$.

NOTATION 6.1. Let $d = n - 1$. Let the pairs of tuples, $\boldsymbol{\mu} \in \mathbb{L}^d$ and $\boldsymbol{\nu} \in \mathbb{C}^d$, $\boldsymbol{\delta} \in (\mathbb{Z}/2\mathbb{Z})^{d+1}$ and $\boldsymbol{\epsilon} \in (\mathbb{Z}/2\mathbb{Z})^d$, $\boldsymbol{m} \in \mathbb{Z}^{d+1}$ and $\boldsymbol{k} \in \mathbb{Z}^d$, be subjected to the following relations

$$\nu_l = \mu_l - \mu_{d+1}, \quad \epsilon_l = \delta_l + \delta_{d+1}, \quad k_l = m_l - m_{d+1},$$

for $l = 1, \dots, d$.

Instead of Hankel transforms, we shall be interested in their Bessel kernels. Therefore, it is convenient to furthermore assume that $\varphi \in \mathcal{S}(\mathbb{R}_+)$ and $\nu \in \mathcal{S}(\mathbb{F}^\times)$. According to (4.19, 4.28), Proposition 4.17 (3) and 4.21 (3), for such Schwartz functions φ and ν ,

$$(6.1) \quad h_{(\mu, \delta)}\varphi(x) = \int_{\mathbb{R}_+} \varphi(y) j_{(\mu, \delta)}(xy) dy, \quad h_{(\mu, m)}\varphi(x) = 2 \int_{\mathbb{R}_+} \varphi(y) j_{(\mu, m)}(xy) y dy,$$

$$(6.2) \quad \mathcal{H}_{(\mu, \delta)}\nu(x) = \int_{\mathbb{R}^\times} \nu(y) J_{(\mu, \delta)}(xy) dy, \quad \mathcal{H}_{(\mu, m)}\nu(z) = \int_{\mathbb{C}^\times} \nu(u) J_{(\mu, m)}(zu) du,$$

with the index $(\boldsymbol{\mu}, \boldsymbol{\delta}) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ or $(\boldsymbol{\mu}, \boldsymbol{m}) \in \mathbb{C}^n \times \mathbb{Z}^n$ being arbitrary.

6.1. The formal integral $J_{\nu, \epsilon}(x, \pm)$. To motivate the definition of $J_{\nu, \epsilon}(x, \pm)$, we shall do certain operations on the Fourier type integral (5.8) in Theorem 5.8. In the meanwhile, we shall forget the assumption $\Re \mu_1 > \dots > \Re \mu_n$, which is required for convergence.

Upon making the change of variables, $x_n = (x_1 \dots x_{n-1})^{-1} xy$, $x_l = |xy|^{\frac{1}{n}} y_l^{-1}$, $l = 1, \dots, n-1$, one turns (5.8) into

$$\begin{aligned} \mathcal{H}_{(\mu, \delta)}\nu(x) = & \int_{\mathbb{R}^\times \times \mathbb{R}^n} \nu(y) \operatorname{sgn}(xy)^{\delta_n} \left(\prod_{l=1}^{n-1} \operatorname{sgn}(y_l)^{\delta_l + \delta_n} |y_l|^{\mu_l - \mu_n - 1} \right) \\ & e \left(|xy|^{\frac{1}{n}} \left(\operatorname{sgn}(xy) \cdot y_1 \dots y_{n-1} + \sum_{l=1}^{n-1} y_l^{-1} \right) \right) dy dy_{n-1} \dots dy_1. \end{aligned}$$

In symbolic notation, moving the integral over dy to the outermost place and comparing the resulted integral with the right hand side of the first formula in (6.2), the Bessel kernel $J_{(\mu,\delta)}(x)$ is then represented by the following formal integral over $dy_{n-1} \dots dy_1$,

$$\text{sgn}(x)^{\delta_n} \int_{\mathbb{R} \times \mathbb{R}^{n-1}} \left(\prod_{l=1}^{n-1} \text{sgn}(y_l)^{\delta_l + \delta_n} |y_l|^{\mu_l - \mu_n - 1} \right) e \left(|x|^{\frac{1}{n}} \left(\text{sgn}(x) \cdot y_1 \dots y_{n-1} + \sum_{l=1}^{n-1} y_l^{-1} \right) \right) dy_{n-1} \dots dy_1.$$

We define the formal integral

$$(6.3) \quad J_{\nu, \epsilon}(x, \pm) = \int_{\mathbb{R} \times \mathbb{R}^d} \left(\prod_{l=1}^d \text{sgn}(y_l)^{\epsilon_l} |y_l|^{\nu_l - 1} \right) e^{ix(\pm y_1 \dots y_d + \sum_{l=1}^d y_l^{-1})} dy_d \dots dy_1, \quad x \in \mathbb{R}_+.$$

Thus, in view of Notation 6.1, we have $J_{(\mu,\delta)}(\pm x) = (\pm)^{\delta_{d+1}} J_{\nu, \epsilon}(2\pi x^{\frac{1}{d+1}}, \pm)$ in symbolic notation.

6.2. The formal integral $j_{\nu, \delta}(x)$. For $\nu \in \mathbb{C}^d$ and $\delta \in (\mathbb{Z}/2\mathbb{Z})^{d+1}$, we define the formal integral

$$(6.4) \quad j_{\nu, \delta}(x) = \int_{\mathbb{R}_+^d} j_{(0, \delta_{d+1})}(xy_1 \dots y_d) \prod_{l=1}^d y_l^{\nu_l - 1} j_{(0, \delta_l)}(xy_l^{-1}) dy_d \dots dy_1, \quad x \in \mathbb{R}_+.$$

One may derive the symbolic identity $j_{(\mu, \delta)}(x) = j_{\nu, \delta}(x^{\frac{1}{d+1}})$ from Corollary 4.16 and 5.9, combined with the first formula in (6.1).

6.3. The integral $J_{\nu, k}(x, u)$. First of all, proceeding in the same way as in §6.1, from the Fourier type integral (5.11) in Theorem 5.10, we can deduce the symbolic equality $J_{(\mu, m)}(xe^{i\phi}) = e^{-im_{d+1}\phi} J_{\nu, k}(2\pi x^{\frac{1}{d+1}}, e^{i\phi})$, with the definition of the formal integral,

$$(6.5) \quad J_{\nu, k}(x, u) = \int_{\mathbb{C} \times \mathbb{R}^d} \left(\prod_{l=1}^d [u_l]^{k_l} \|u_l\|^{\nu_l - 1} \right) e^{ix\Lambda(uu_1 \dots u_d + \sum_{l=1}^d u_l^{-1})} du_d \dots du_1, \\ x \in \mathbb{R}_+, u \in \mathbb{C}, |u| = 1.$$

Here, we recall that $\Lambda(z) = z + \bar{z}$.

In the polar coordinates, we write $u_l = y_l e^{i\theta_l}$ and $u = e^{i\phi}$. Moving the integral over the torus $(\mathbb{R}/2\pi\mathbb{Z})^d$ inside, in symbolic manner, the integral above turns into

$$2^d \int_{\mathbb{R}_+^d} \int_{(\mathbb{R}/2\pi\mathbb{Z})^d} \left(\prod_{l=1}^d y_l^{2\nu_l - 1} \right) e^{i \sum_{l=1}^d k_l \theta_l} e^{2ix(y_1 \dots y_d \cos(\sum_{l=1}^d \theta_l + \phi) + \sum_{l=1}^d y_l^{-1} \cos \theta_l)} d\theta_d \dots d\theta_1 dy_d \dots dy_1.$$

Let us introduce the following definitions

$$(6.6) \quad \Theta_k(\theta, y; x, \phi) = 2xy_1 \dots y_d \cos \left(\sum_{l=1}^d \theta_l + \phi \right) + \sum_{l=1}^d (k_l \theta_l + 2xy_l^{-1} \cos \theta_l),$$

$$(6.7) \quad J_k(\mathbf{y}; x, \phi) = \int_{(\mathbb{R}/2\pi\mathbb{Z})^d} e^{i\Theta_k(\theta; \mathbf{y}; x, \phi)} d\theta,$$

$$(6.8) \quad p_{2\nu}(\mathbf{y}) = \prod_{l=1}^d y_l^{2\nu_l-1},$$

with $\mathbf{y} = (y_1, \dots, y_d)$, $\theta = (\theta_1, \dots, \theta_d)$. Then (6.5) can be symbolically rewritten as

$$(6.9) \quad J_{\nu, k}(x, e^{i\phi}) = 2^d \int_{\mathbb{R}_+^d} p_{2\nu}(\mathbf{y}) J_k(\mathbf{y}; x, \phi) d\mathbf{y}, \quad x \in \mathbb{R}_+, \phi \in \mathbb{R}/2\pi\mathbb{Z}.$$

THEOREM 6.2. *Let $(\mu, \mathbf{m}) \in \mathbb{L}^d \times \mathbb{Z}^{d+1}$ and $(\nu, \mathbf{k}) \in \mathbb{C}^d \times \mathbb{Z}^d$ satisfy the relations given in Notation 6.1. Suppose $\nu \in \bigcup_{a \in [-\frac{1}{2}, 0]} \{\nu \in \mathbb{C}^d : -\frac{1}{2} < 2\Re \nu_l + a < 0 \text{ for all } l = 1, \dots, d\}$.*

(1). *The integral in (6.9) converges absolutely. Subsequently, we shall therefore use (6.9) as the definition of $J_{\nu, k}(x, e^{i\phi})$.*

(2). *We have the (genuine) identity*

$$J_{(\mu, \mathbf{m})}(xe^{i\phi}) = e^{-im_{d+1}\phi} J_{\nu, k}(2\pi x^{\frac{1}{d+1}}, e^{i\phi}).$$

6.4. The integral $j_{\nu, m}(x)$. Let us consider the integral $j_{\nu, m}(x)$ defined by

$$(6.10) \quad j_{\nu, m}(x) = 2^d \int_{\mathbb{R}_+^d} j_{(0, m_{d+1})}(xy_1 \dots y_d) \prod_{l=1}^d y_l^{2\nu_l-1} j_{(0, m_l)}(xy_l^{-1}) dy_d \dots dy_1,$$

with $\nu \in \mathbb{C}^d$ and $\mathbf{m} \in \mathbb{Z}^{d+1}$.

6.4.1. *Absolute convergence of $j_{\nu, m}(x)$.* In contrary to the real case, where the integral $j_{\nu, \delta}(x)$ never absolutely converges, $j_{\nu, m}(x)$ is actually absolutely convergent, if each component of ν lies in a certain vertical strip of width at least $\frac{1}{4}$.

DEFINITION 6.3. *For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ such that $a_l < b_l$ for all $l = 1, \dots, d$, we define the open hyper-strip $\mathbb{S}^d(\mathbf{a}, \mathbf{b}) = \{\nu \in \mathbb{C}^d : \Re \nu_l \in (a_l, b_l)\}$. We simply denote $\mathbb{S}^d(a, b) = \mathbb{S}^d(ae^d, be^d)$.*

PROPOSITION 6.4. *Let $(\nu, \mathbf{m}) \in \mathbb{C}^d \times \mathbb{Z}^{d+1}$. The integral $j_{\nu, m}(x)$ defined above by (6.10) absolutely converges if $\nu \in \bigcup_{a \in [-\frac{1}{2}, |m_{d+1}|]} \mathbb{S}^d(\frac{1}{2}(-\frac{1}{2} - a)e^d, \frac{1}{2}(\|\mathbf{m}^d\| - ae^d))$, with $\mathbf{m}^d = (m_1, \dots, m_d)$ and $\|\mathbf{m}^d\| = (|m_1|, \dots, |m_d|)$.*

To show this, we first recollect some well-known facts concerning $J_m(x)$, as $j_{(0, m)}(x) = 2\pi i^m J_m(4\pi x)$ in view of (4.29).

Firstly, for $m \in \mathbb{N}$, we have the Poisson-Lommel integral representation (see [Wat, 3.3 (1)])

$$(6.11) \quad J_m(x) = \frac{\left(\frac{1}{2}x\right)^m}{\Gamma(m + \frac{1}{2}) \Gamma(\frac{1}{2})} \int_0^\pi \cos(x \cos \theta) \sin^{2m} \theta d\theta.$$

This yields the bound

$$(6.12) \quad |J_m(x)| \leq \frac{\sqrt{\pi} \left(\frac{1}{2}x\right)^{|m|}}{\Gamma(|m| + \frac{1}{2})},$$

for $m \in \mathbb{Z}$. Secondly, the asymptotic expansion of $J_m(x)$ (see [Wat, 7.21 (1)]) provides the estimate

$$(6.13) \quad J_m(x) \ll_m x^{-\frac{1}{2}}.$$

Combining these, we then arrive at the following lemma.

LEMMA 6.5. *Let m be an integer.*

(1). *We have the estimates*

$$j_{(0,m)}(x) \ll_m x^{|m|}, \quad j_{(0,m)}(x) \ll_m x^{-\frac{1}{2}}.$$

(2). *More generally, for any $a \in [-\frac{1}{2}, |m|]$, we have the estimate*

$$j_{(0,m)}(x) \ll_m x^a.$$

PROOF OF PROPOSITION 6.4. We divide $\mathbb{R}_+ = (0, \infty)$ into the union of two intervals, $I_- \cup I_+ = (0, 1] \cup [1, \infty)$. Accordingly, the integral in (6.10) is partitioned into 2^d many integrals, each of which is supported on some hyper-cube $I_{\mathbf{Q}} = I_{\mathbf{Q}_1} \times \dots \times I_{\mathbf{Q}_d}$ for $\mathbf{Q} \in \{+, -\}^d$. For each such integral, we estimate $j_{(0,m_l)}(xy_l^{-1})$ using the first or the second estimate in Lemma 6.5 (1) according as $\mathbf{Q}_l = +$ or $\mathbf{Q}_l = -$ and apply the bound in Lemma 6.5 (2) for $j_{(0,m_{d+1})}(xy_1 \dots y_d)$. In this way, for any $a \in [-\frac{1}{2}, |m_{d+1}|]$, one has

$$\begin{aligned} & 2^d \int_{\mathbb{R}_+^d} |j_{(0,m_{d+1})}(xy_1 \dots y_d)| \prod_{l=1}^d |y_l^{2\nu_l-1} j_{(0,m_l)}(xy_l^{-1})| dy_d \dots dy_1 \\ & \ll \sum_{\mathbf{Q} \in \{+, -\}^d} x^{\sum_{l \in L_+(\mathbf{Q})} |m_l| - \frac{1}{2} |L_-(\mathbf{Q})| + a} I_{2\nu+a\mathbf{e}^d, \mathbf{m}^d}(\mathbf{Q}), \end{aligned}$$

with the auxiliary definition

$$I_{\lambda, \mathbf{k}}(\mathbf{Q}) = \int_{I_{\mathbf{Q}}} \left(\prod_{l \in L_+(\mathbf{Q})} y_l^{\Re \lambda_l - |k_l| - 1} \right) \left(\prod_{l \in L_-(\mathbf{Q})} y_l^{\Re \lambda_l - \frac{1}{2}} \right) dy_d \dots dy_1, \quad (\lambda, \mathbf{k}) \in \mathbb{C}^d \times \mathbb{Z}^d,$$

and $L_{\pm}(\mathbf{Q}) = \{l : \mathbf{Q}_l = \pm\}$. The implied constant depends only on \mathbf{m} and d . It is clear that all the integrals $I_{2\nu+a\mathbf{e}^d, \mathbf{m}^d}(\mathbf{Q})$ absolutely converge if $-\frac{1}{2} < 2\Re \nu_l + a < |m_l|$ for all $l = 1, \dots, d$. The proof is then completed. Q.E.D.

REMARK 6.6. *When $d = 1$, one may apply the two estimates in Lemma 6.5 (1) to $j_{(0,m_2)}(xy)$ in the similar fashion as $j_{(0,m_1)}(xy^{-1})$. Then*

$$\begin{aligned} & 2 \int_{\mathbb{R}_+} |y^{2\nu-1} j_{(0,m_1)}(xy^{-1}) j_{(0,m_2)}(xy)| dy \\ & \ll_{m_1, m_2} x^{|m_1| - \frac{1}{2}} \int_1^\infty y^{2\Re \nu - |m_1| - \frac{3}{2}} dy + x^{|m_2| - \frac{1}{2}} \int_0^1 y^{2\Re \nu + |m_2| - \frac{1}{2}} dy. \end{aligned}$$

Since both integrals above absolutely converge if $-|m_2| - \frac{1}{2} < 2\Re \nu < |m_1| + \frac{1}{2}$, this also proves Proposition 6.4 in the case $d = 1$.

6.4.2. Equality between $j_{(\mu, m)}(x)$ and $j_{\nu, m}(x^{\frac{1}{n}})$.

PROPOSITION 6.7. Let $(\nu, m) \in \mathbb{C}^d \times \mathbb{Z}^{d+1}$ be as in Proposition 6.4 so that the integral $j_{\nu, m}(x)$ absolutely converges. Suppose that $\mu \in \mathbb{L}^d$ and $\nu \in \mathbb{C}^d$ satisfy the relations given in Notation 6.1. Then we have the identity

$$j_{(\mu, m)}(x) = j_{\nu, m}(x^{\frac{1}{n}}).$$

PROOF. Some change of variables turns the integral in Corollary 5.11 into

$$2^{d+1} e^{im\phi} \int_{\mathbb{R}_+^{d+1}} \varphi(y) j_{(0, m_{d+1})}((xy)^{\frac{1}{d+1}} y_1 \dots y_d) \prod_{l=1}^d y_l^{2\nu_l-1} j_{(0, m_l)}((xy)^{\frac{1}{d+1}} y_l^{-1}) y dy dy_d \dots dy_1.$$

Corollary 4.20 and 5.11, along with the second formula in (6.1), yield

$$2 \int_{\mathbb{R}_+} \varphi(y) j_{(\mu, m)}(xy) y dy = 2^{d+1} \int_{\mathbb{R}_+^{d+1}} \varphi(y) j_{(0, m_{d+1})}((xy)^{\frac{1}{d+1}} y_1 \dots y_d) \prod_{l=1}^d y_l^{2\nu_l-1} j_{(0, m_l)}((xy)^{\frac{1}{d+1}} y_l^{-1}) y dy dy_d \dots dy_1,$$

for any $\varphi \in \mathcal{S}(\mathbb{R}_+)$, provided that $\Re \mu_1 > \dots > \Re \mu_{d+1}$ or equivalently $\Re \nu_1 > \dots > \Re \nu_d > 0$. In view of Proposition 6.4, the integral on the right hand side is absolute convergent at least when $\frac{1}{4} > \Re \nu_1 > \dots > \Re \nu_d > 0$. Therefore, the asserted equality holds on the domain $\{\nu \in \mathbb{C}^d : \frac{1}{4} > \Re \nu_1 > \dots > \Re \nu_d > 0\}$ and remains valid on the whole domain of convergence of $j_{\nu, m}(x)$ given in Proposition 6.4 due to the principle of analytic continuation. Q.E.D.

6.4.3. An auxiliary lemma.

LEMMA 6.8. Let $(\nu, m) \in \mathbb{C}^d \times \mathbb{Z}^{d+1}$ and $m \in \mathbb{Z}$. Set $A = \max_{l=1, \dots, d+1} \{|m_l|\}$. Suppose $\nu \in \bigcup_{a \in [-\frac{1}{2}, 0]} \mathbb{S}^d(-\frac{1}{4} - \frac{1}{2}a, -\frac{1}{2}a)$. We have the estimate

$$\begin{aligned} & 2^d \int_{\mathbb{R}_+^d} |j_{(0, m_{d+1}+m)}(xy_1 \dots y_d)| \prod_{l=1}^d |y_l^{2\nu_l-1} j_{(0, m_l+m)}(xy_l^{-1})| dy_d \dots dy_1 \\ & \ll \sum_{\varrho \neq \varrho^-} \left(\frac{2\pi e x}{|m|+1} \right)^{|L_+(\varrho)|+|m|} (|m|+1)^{2|L_-(\varrho)|+A|L_+(\varrho)|} \\ & \quad x^{-\frac{1}{2}|L_-(\varrho)|} \max \left\{ x^{|L_+(\varrho)|A}, x^{-|L_+(\varrho)|A-\frac{1}{2}} \right\} \\ & \quad + \left(\frac{2\pi e x}{|m|+1} \right)^{|m|} (|m|+1)^A x^{-\frac{d}{2}} \max \{x^A, x^{-A}\}, \end{aligned}$$

where $\varrho \in \{+, -\}^d$, $\varrho^- = (-, \dots, -)$ and $L_{\pm}(\varrho) = \{l : \varrho_l = \pm\}$. The implied constant depends only on m and d .

Firstly, we require the bound (6.12) for $J_m(x)$. Secondly, we observe that when $x \geq (|m|+1)^2$ the bound (6.13) for $J_m(x)$ can be improved so that the implied constant becomes

absolute. This follows from the asymptotic expansion of $J_m(x)$ given in [Olv, §7.13.1]. Moreover, we have Bessel's integral representation (see [Wat, 2.2 (1)])

$$(6.14) \quad J_m(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(m\theta - x \sin \theta) d\theta,$$

which yields the bound

$$(6.15) \quad |J_m(x)| \leq 1.$$

We then have the following lemma (compare [HM, Proposition 8]).

LEMMA 6.9. *Let m be an integer.*

(1). *The following two estimates hold*

$$j_{(0,m)}(x) \ll \frac{(2\pi x)^{|m|}}{\Gamma(|m| + \frac{1}{2})}, \quad j_{(0,m)}(x) \ll \frac{|m| + 1}{\sqrt{x}},$$

with absolute implied constants.

(2). *For any $a \in [-\frac{1}{2}, 0]$ we have the estimate*

$$j_{(0,m)}(x) \ll (|m| + 1)^{-2} x^a,$$

with absolute implied constants.

PROOF OF LEMMA 6.8. Our proof here is similar to that of Proposition 6.4, except that

- Lemma 6.9 (1) and (2) are applied in place of Lemma 6.5 (1) and (2) respectively to bound $j_{(0,m_l+m)}(xy_l^{-1})$ and $j_{(0,m_{d+1}+m)}(xy_1 \dots y_d)$, and
- the first estimate in Lemma 6.9 (1) is used for $j_{(0,m_{d+1}+m)}(xy_1 \dots y_d)$ in the case $\varrho = \varrho^-$.

In this way, one obtains the following estimate

$$\begin{aligned} & 2^d \int_{\mathbb{R}_+^d} |j_{(0,m_{d+1}+m)}(xy_1 \dots y_d)| \prod_{l=1}^d |y_l^{2\nu_l-1} j_{(0,m_l+m)}(xy_l^{-1})| dy_d \dots dy_1 \\ & \ll \sum_{\varrho \neq \varrho^-} \frac{\prod_{l \in L_-(\varrho)} (|m_l + m| + 1)^{1-2a}}{\prod_{l \in L_+(\varrho)} \Gamma(|m_l + m| + \frac{1}{2}) (|m_l + m| + 1)^{2a}} \\ & \quad (2\pi x)^{\sum_{l \in L_+(\varrho)} |m_l + m| - \frac{1}{2}|L_-(\varrho)| + a} I_{2\nu + ae^d, \mathbf{m}^d + me^d}(\varrho) \\ & \quad + \frac{\prod_{l=1}^d (|m_l + m| + 1)}{\Gamma(|m_{d+1} + m| + \frac{1}{2})} (2\pi x)^{-\frac{d}{2} + |m_{d+1} + m|} I_{2\nu + |m_{d+1} + m|e^d}(\varrho^-), \end{aligned}$$

^{IX}with $a \in [-\frac{1}{2}, 0]$. Now the implied constant above depends only on d . Suppose that $-\frac{1}{2} - a < 2 \Re \nu_l < -a$ for all $l = 1, \dots, d$, then all the integrals $I_{2\nu + ae^d, \mathbf{m}^d + me^d}(\varrho)$ and $I_{2\nu + |m_{d+1} + m|e^d}(\varrho^-)$ absolutely converges, and are of size $O_d\left(\prod_{l \in L_+(\varrho)} (|m_l + m| + 1)^{-1}\right)$ and $O_d\left((|m_{d+1} + m| + 1)^{-d}\right)$ respectively. A final estimation using Stirling's asymptotic formula yields our asserted bound. Q.E.D.

^{IX}When $\varrho = \varrho^-$, k does not occur in the definition of $I_{\mathbf{k},k}(\varrho^-)$ and is therefore suppressed from the subscript.

REMARK 6.10. In the case $d = 1$, modifying over the ideas in Remark 6.6, one may show the slightly improved estimate

$$2 \int_{\mathbb{R}_+} |y^{2\nu-1} j_{(0,m_1+m)}(xy^{-1}) j_{(0,m_2+m)}(xy)| dy \\ \ll_{m_1,m_2} \left(\frac{2\pi e x}{|m|+1} \right)^{|m|} (|m|+1)^A x^{-\frac{1}{2}} \max \{x^A, x^{-A}\},$$

given that $|\Re \nu| < \frac{1}{4}$, with $A = \max \{|m_1|, |m_2|\}$.

6.5. The series of integrals $J_{\nu,m}(x, u)$. We define the following series of integrals,

$$(6.16) \quad J_{\nu,m}(x, u) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} u^m j_{\nu,m+me^n}(x) = \\ = \frac{2^{d-1}}{\pi} \sum_{m \in \mathbb{Z}} u^m \int_{\mathbb{R}_+^d} j_{(0,m_{d+1}+m)}(xy_1 \dots y_d) \prod_{l=1}^d y_l^{2\nu_l-1} j_{(0,m_l+m)}(xy_l^{-1}) dy_d \dots dy_1,$$

with $x \in \mathbb{R}_+$ and $u \in \mathbb{C}$, $|u| = 1$.

6.5.1. *Absolute convergence of $J_{\nu,m}(x, u)$.* We have the following simple consequence of Lemma 6.8.

PROPOSITION 6.11. Let $(\nu, m) \in \mathbb{C}^d \times \mathbb{Z}^{d+1}$. The series of integrals $J_{\nu,m}(x, u)$ defined by (6.16) is absolutely convergent if $\nu \in \bigcup_{a \in [-\frac{1}{2}, 0]} \mathbb{S}^d(-\frac{1}{4} - \frac{1}{2}a, -\frac{1}{2}a)$.

6.5.2. *Equality between $J_{(\mu,m)}(xe^{i\phi})$ and $J_{\nu,m}(x^{\frac{1}{n}}, e^{i\phi})$.* In view of Proposition 6.7 along with (4.45) and (6.16), the following proposition is readily established.

PROPOSITION 6.12. Let $(\nu, m) \in \mathbb{C}^d \times \mathbb{Z}^{d+1}$. Suppose that ν satisfies the condition in Proposition 6.11 so that $J_{\nu,m}(x, u)$ is absolutely convergent. Then, given that μ and ν satisfy the relations in Notation 6.1, we have the identity

$$J_{(\mu,m)}(xe^{i\phi}) = J_{\nu,m}(x^{\frac{1}{n}}, e^{i\phi}),$$

with $x \in \mathbb{R}_+$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$.

6.6. Proof of Theorem 6.2.

LEMMA 6.13. Let $k \in \mathbb{C}^d$ and recall the integral $J_k(\mathbf{y}; x, \phi)$ defined by (6.6, 6.7). We have the following absolutely convergent series expansion of $J_k(\mathbf{y}; x, \phi)$

$$(6.17) \quad J_k(\mathbf{y}; 2\pi x, \phi) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{im\phi} j_{(0,m)}(xy_1 \dots y_d) \prod_{l=1}^d j_{(0,k_l+m)}(xy_l^{-1}).$$

PROOF. In view of Example 4.22, we have the integral representation

$$j_{(0,m)}(x) = \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{im\theta + 4\pi i x \cos \theta} d\theta$$

as well as the Fourier series expansion

$$e^{4\pi i x \cos \phi} = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} j_{(0,m)}(x) e^{im\phi}.$$

Therefore

$$\begin{aligned}
& \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{im\phi} j_{(0,m)}(xy_1 \dots y_d) \prod_{l=1}^d j_{(0,k_l+m)}(xy_l^{-1}) \\
&= \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{im\phi} j_{(0,m)}(xy_1 \dots y_d) \int_{(\mathbb{R}/2\pi\mathbb{Z})^d} e^{im \sum_{l=1}^d \theta_l} e^{i \sum_{l=1}^d (ik_l \theta + 4\pi i xy_l^{-1} \cos \theta_l)} d\theta_d \dots d\theta_1 \\
&= \int_{(\mathbb{R}/2\pi\mathbb{Z})^d} \left(\frac{1}{2\pi} \sum_{m \in \mathbb{Z}} e^{im(\sum_{l=1}^d \theta_l + \phi)} j_{(0,m)}(xy_1 \dots y_d) \right) e^{i \sum_{l=1}^d (ik_l \theta + 4\pi i xy_l^{-1} \cos \theta_l)} d\theta_d \dots d\theta_1 \\
&= \int_{(\mathbb{R}/2\pi\mathbb{Z})^d} e^{4\pi i xy_1 \dots y_d \cos(\sum_{l=1}^d \theta_l + \phi)} e^{i \sum_{l=1}^d (ik_l \theta + 4\pi i xy_l^{-1} \cos \theta_l)} d\theta_d \dots d\theta_1.
\end{aligned}$$

The absolute convergence required for the validity of each equality above is justified by the first estimate of $j_{(0,m)}(x)$ in Lemma 6.9 (1). The proof is completed, since the last line is exactly the definition of $J_k(y; 2\pi x, \phi)$. Q.E.D.

Inserting the series expansion of $J_k(y; 2\pi x, \phi)$ in Lemma 6.13 into the integral in (6.9) and interchanging the order of integration and summation, one arrives exactly at the series of integrals $J_{\nu, (k,0)}(x, e^{i\phi}) = e^{-im_{d+1}\phi} J_{\nu, m}(x, e^{i\phi})$. The first assertion on absolute convergence in Theorem 6.2 follows immediately from Proposition 6.11, whereas the identity in the second assertion is a direct consequence of Proposition 6.12.

6.7. The rank-two case ($d = 1$).

6.7.1. *The real case.* The formal integral representation $J_{\nu, \epsilon}(2\pi \sqrt{x}, \pm)$ of the Bessel kernel $J_{(\frac{1}{2}\nu, -\frac{1}{2}\nu), (\epsilon, 0)}(\pm x)$ is reduced to the following integral representations of classical Bessel functions

$$\pm \pi i e^{\pm \frac{1}{2}\pi i \nu} H_{\nu}^{(1,2)}(2x) = \int_0^{\infty} y^{\nu-1} e^{\pm i x(y+y^{-1})} dy, \quad 2e^{\pm \frac{1}{2}\pi i \nu} K_{\nu}(2x) = \int_0^{\infty} y^{\nu-1} e^{\pm i x(y-y^{-1})} dy,$$

which are only (conditionally) convergent when $|\Re \nu| < 1$ (see [Qi1, §2.3.2]).

6.7.2. The complex case.

LEMMA 6.14. *Let $k \in \mathbb{Z}$. Recall from (6.6, 6.7) the definition*

$$J_k(y; x, \phi) = \int_0^{2\pi} e^{ik\theta + 2ixy^{-1} \cos \theta + 2ixy \cos(\theta + \phi)} d\theta, \quad x, y \in (0, \infty), \phi \in [0, 2\pi).$$

Define $Y(y, \phi) = |y^{-1} + ye^{i\phi}| = \sqrt{y^{-2} + 2 \cos \phi + y^2}$, $\Phi(y, \phi) = \arg(y^{-1} + ye^{i\phi})$ and $E(y, \phi) = e^{i\Phi(y, \phi)}$. Then

$$(6.18) \quad J_k(y; x, \phi) = 2\pi i^k E(y, \phi)^{-k} J_k(2xY(y, \phi)).$$

PROOF. (6.18) follows immediately from the identity

$$2\pi i^k J_k(x) = \int_0^{2\pi} e^{ik\theta + ix \cos \theta} d\theta,$$

along with the observation

$$y^{-1} \cos \theta + y \cos(\theta + \phi) = \Re \left(y^{-1} e^{i\theta} + ye^{i(\theta + \phi)} \right) = Y(y, \phi) \cos(\theta + \Phi(y, \phi)).$$

Q.E.D.

PROPOSITION 6.15. *Let $\nu \in \mathbb{C}$ and $k \in \mathbb{Z}$. Recall the definition of $J_{\nu,k}(x, e^{i\phi})$ given by (6.9). Then*

$$(6.19) \quad J_{\nu,k}(x, e^{i\phi}) = 4\pi i^k \int_0^\infty y^{2\nu-1} [y^{-1} + ye^{i\phi}]^{-k} J_k(2x|y^{-1} + ye^{i\phi}|) dy,$$

with $x \in (0, \infty)$ and $\phi \in [0, 2\pi)$. Here, we recall the notation $[z] = z/|z|$. The integral in (6.19) converges when $|\Re \nu| < \frac{3}{4}$ and the convergence is absolute if and only if $|\Re \nu| < \frac{1}{4}$. Moreover, it is analytic with respect to ν on the open vertical strip $\mathbb{S}(-\frac{3}{4}, \frac{3}{4})$.

PROOF. (6.19) follows immediately from Lemma 6.14.

As for the convergence, since one arrives at an integral of the same form with ν, ϕ replaced by $-\nu, -\phi$ if the variable is changed from y to y^{-1} , it suffices to consider the integral

$$\int_2^\infty y^{2\nu-1} e^{-ik\Phi(y,\phi)} J_k(2xY(y,\phi)) dy,$$

for $\Re \nu < \frac{3}{4}$. We have the following asymptotic of $J_k(x)$ (see [Wat, 7.21 (1)])

$$J_k(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x - \frac{1}{2}k\pi - \frac{1}{4}\pi\right) + O_k(x^{-\frac{3}{2}}).$$

The error term contributes an absolutely convergent integral when $\Re \nu < \frac{3}{4}$, whereas the integral coming from the main term absolutely converges if and only if $\Re \nu < \frac{1}{4}$. We are now reduced to the integral

$$\int_2^\infty y^{2\nu-1} e^{-ik\Phi(y,\phi)} (xY(y,\phi))^{-\frac{1}{2}} e^{\pm 2ixY(y,\phi)} dy.$$

In order to see the convergence, we split out $e^{\pm 2ixy}$ from $e^{\pm 2ixY(y,\phi)}$ and put $f_{\nu,k}(y; x, \phi) = y^{2\nu-1} e^{-ik\Phi(y,\phi)} (xY(y,\phi))^{-\frac{1}{2}} e^{\pm 2ix(Y(y,\phi)-y)}$. Partial integration turns the above integral into

$$\mp \frac{1}{2ix^{\frac{3}{2}}} \left(2^{2\nu-1} e^{-ik\Phi(2,\phi)} Y(2,\phi)^{-\frac{1}{2}} e^{\pm 2ixY(2,\phi)} + \int_2^\infty (\partial f_{\nu,k}/\partial y)(y; x, \phi) e^{\pm 2ixy} dy \right).$$

Some calculations show that $(\partial f_{\nu,k}/\partial y)(y; x, \phi) \ll_{\nu,k,x} y^{2\Re \nu - \frac{3}{2}}$ for $y \geq 2$, and hence the integral in the second term is absolutely convergent when $\Re \nu < \frac{3}{4}$. With the above arguments, the analyticity with respect to ν is obvious. Q.E.D.

COROLLARY 6.16. *Let $\mu \in \mathbb{S}(-\frac{3}{8}, \frac{3}{8})$ and $m \in \mathbb{Z}$. We have*

$$(6.20) \quad J_{(\mu, -\mu, m, 0)}(xe^{i\phi}) = 4\pi i^m \int_0^\infty y^{4\mu-1} [y^{-1} + ye^{i\phi}]^{-m} J_m(4\pi\sqrt{x}|y^{-1} + ye^{i\phi}|) dy,$$

with $x \in (0, \infty)$ and $\phi \in [0, 2\pi)$. The integral in (6.20) converges if $|\Re \mu| < \frac{3}{8}$ and absolutely converges if and only if $|\Re \mu| < \frac{1}{8}$.

PROOF. From Theorem 6.2, we see that (6.20) holds for $\mathbb{S}(-\frac{1}{8}, \frac{1}{8})$. In view of Proposition 6.15, the right hand side of (6.20) is analytic in μ on $\mathbb{S}(-\frac{3}{8}, \frac{3}{8})$, and therefore it is allowed to extend the domain of equality from $\mathbb{S}(-\frac{1}{8}, \frac{1}{8})$ onto $\mathbb{S}(-\frac{3}{8}, \frac{3}{8})$. Q.E.D.

7. Two connection formulae for $J_{(\mu, \mathbf{m})}(z)$

In this section, we shall prove two formulae for $J_{(\mu, \mathbf{m})}(z)$ in connection with the two kinds of Bessel functions of rank n and positive sign. These Bessel functions arise as solutions of Bessel equations in [Qi1, §7] and their relations are made clear in [Qi1, §8.2]. Our motivation is based on the following self-evident identity for the rank-one example

$$e(z + \bar{z}) = e(z)e(\bar{z}).$$

7.1. The first connection formula. For $\varsigma \in \{+, -\}$, $\lambda \in \mathbb{C}^n$ and $l = 1, \dots, n$, we define the following series of ascending powers of z (see [Qi1, §7.1])

$$(7.1) \quad J_l(z; \varsigma, \lambda) = \sum_{m=0}^{\infty} \frac{(\varsigma^n)^m z^{n(-\lambda_l + m)}}{\prod_{k=1}^n \Gamma(\lambda_k - \lambda_l + m + 1)}, \quad z \in \mathbb{U}.$$

$J_l(z; \varsigma, \lambda)$ is called a *Bessel function of the first kind*, n , ς and λ its rank, sign and index, respectively. Since the definition (7.1) is valid for any $\lambda \in \mathbb{C}^n$, the assumption $\lambda \in \mathbb{L}^{n-1}$ that we imposed in [Qi1] is rather superfluous. Also, we have the following formula in the same fashion as (4.6) in Lemma 4.6,

$$(7.2) \quad J_l(z; \varsigma, \lambda - \lambda \mathbf{e}^n) = z^{n\lambda} J_l(z; \varsigma, \lambda).$$

THEOREM 7.1. *Let $(\mu, \mathbf{m}) \in \mathbb{L}^{n-1} \times \mathbb{Z}^n$. We have*

$$(7.3) \quad J_{(\mu, \mathbf{m})}(z) = (2\pi^2)^{n-1} \sum_{l=1}^n S_l(\mu, \mathbf{m}) J_l(2\pi z^{\frac{1}{n}}; +, \mu + \frac{1}{2}\mathbf{m}) J_l(2\pi \bar{z}^{\frac{1}{n}}; +, \mu - \frac{1}{2}\mathbf{m}),$$

with $S_l(\mu, \mathbf{m}) = \prod_{k \neq l} (\pm i)^{m_l - m_k} \sin(\pi(\mu_l - \mu_k \pm \frac{1}{2}(m_l - m_k)))^{-1}$. Here, $z^{\frac{1}{n}}$ is the principal branch of the n -th root of z , that is $(xe^{i\phi})^{\frac{1}{n}} = x^{\frac{1}{n}} e^{\frac{1}{n}i\phi}$. The expression on the right hand side of (7.3) is independent on the choice of the argument of z modulo 2π . It is understood that the right hand side should be replaced by its limit if (μ, \mathbf{m}) is not generic with respect to the order \leq on $\mathbb{C} \times \mathbb{Z}$ in the sense of Definition 4.1.

PROOF. Recall from (2.6, 2.7, 4.22, 4.45) that

$$J_{(\mu, \mathbf{m})}(xe^{i\phi}) = (2\pi)^{n-1} \sum_{m=-\infty}^{\infty} i^{\|\mathbf{m} + m\mathbf{e}^n\|} e^{im\phi} \frac{1}{2\pi i} \int_{\mathcal{C}_{(\mu, \mathbf{m} + m\mathbf{e}^n)}} \left(\prod_{l=1}^n \frac{\Gamma(s - \mu_l + \frac{1}{2}|m_l + m|)}{\Gamma(1 - s + \mu_l + \frac{1}{2}|m_l + m|)} \right) ((2\pi)^n x)^{-2s} ds.$$

Assume first that (μ, \mathbf{m}) is generic with respect to the order \leq on $\mathbb{C} \times \mathbb{Z}$. The sets of poles of the gamma factors in the above integral are $\{\mu_l - \frac{1}{2}|m_l + m| - \alpha\}_{\alpha \in \mathbb{N}, l=1, \dots, n}$. With the generic assumption, the integrand has only *simple* poles. We left shift the integral contour of each integral in the series and pick up the residues from these poles. The contribution from the residues at the poles of the l -th gamma factor is the following absolutely

convergent double series,

$$(2\pi)^{n-1} \sum_{m=-\infty}^{\infty} i^{\sum_{k=1}^n |m_k+m|} e^{im\phi} \sum_{\alpha=0}^{\infty} \frac{(-)^{\alpha} ((2\pi)^n x)^{-2\mu_l+|m_l+m|+2\alpha}}{\alpha! (\alpha + |m_l+m|)!} \prod_{k \neq l} \frac{\Gamma(\mu_l - \mu_k - \frac{1}{2}(|m_l+m| - |m_k+m|) - \alpha)}{\Gamma(1 - \mu_l + \mu_k + \frac{1}{2}(|m_l+m| + |m_k+m|) + \alpha)}.$$

Euler's reflection formula of the Gamma function turns this into

$$\frac{(2\pi^2)^{n-1}}{\prod_{k \neq l} (\pm i)^{m_l-m_k} \sin(\pi(\mu_l - \mu_k \mp \frac{1}{2}(m_l - m_k)))} \sum_{m=-\infty}^{\infty} i^{n|m_l+m|} e^{im\phi} \sum_{\alpha=0}^{\infty} \frac{(-)^{n\alpha} ((2\pi)^n x)^{-2\mu_l+|m_l+m|+2\alpha}}{\prod_{k=1}^n \prod_{\pm} \Gamma(1 - \mu_l + \mu_k + \frac{1}{2}(|m_l+m| \pm |m_k+m|) + \alpha)}.$$

We now interchange the order of summations, truncate the sum over m between $-m_l$ and $-m_l + 1$ and make the change of indices $\beta = \alpha + |m_l+m|$. With the observation that, no matter what m_k is, one of $\frac{1}{2}(|m_l+m| + |m_k+m|)$ and $\frac{1}{2}(|m_l+m| - |m_k+m|)$ is equal to $\frac{1}{2}(m_l - m_k)$ and the other to $|m_l+m| - \frac{1}{2}(m_l - m_k)$ if $m \geq -m_l + 1$, and the signs in front of the two $\frac{1}{2}(m_l - m_k)$ are changed if $m \leq -m_l$, the double series in the expression above turns into

$$\sum_{\alpha=0}^{\infty} \sum_{\beta=\alpha+1}^{\infty} \frac{i^{n(\alpha+\beta)} e^{i(\beta-\alpha-m_l)\phi} ((2\pi)^n x)^{-2\mu_l+\alpha+\beta}}{\prod_{k=1}^n \Gamma(1 - \mu_l + \mu_k + \frac{1}{2}(m_l - m_k) + \alpha) \Gamma(1 - \mu_l + \mu_k - \frac{1}{2}(m_l - m_k) + \beta)} + \sum_{\alpha=0}^{\infty} \sum_{\beta=\alpha}^{\infty} \frac{i^{n(\alpha+\beta)} e^{i(\alpha-\beta-m_l)\phi} ((2\pi)^n x)^{-2\mu_l+\alpha+\beta}}{\prod_{k=1}^n \Gamma(1 - \mu_l + \mu_k - \frac{1}{2}(m_l - m_k) + \alpha) \Gamma(1 - \mu_l + \mu_k + \frac{1}{2}(m_l - m_k) + \beta)},$$

which is then equal to

$$\sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{i^{n(\alpha+\beta)} e^{i(\beta-\alpha-m_l)\phi} ((2\pi)^n x)^{-2\mu_l+\alpha+\beta}}{\prod_{k=1}^n \Gamma(1 - \mu_l + \mu_k + \frac{1}{2}(m_l - m_k) + \alpha) \Gamma(1 - \mu_l + \mu_k - \frac{1}{2}(m_l - m_k) + \beta)}.$$

This double series is clearly independent on the choice of ϕ modulo 2π , and splits exactly as the product

$$J_l(2\pi x^{\frac{1}{n}} e^{\frac{1}{n}i\phi}; +, \mu + \frac{1}{2}\mathbf{m}) J_l(2\pi x^{\frac{1}{n}} e^{-\frac{1}{n}i\phi}; +, \mu - \frac{1}{2}\mathbf{m}).$$

This proves (7.3) in the case when (μ, \mathbf{m}) is generic. As for the nongeneric case, one just passes to the limit. Q.E.D.

7.2. The second connection formula. According to [Qi1, §7.3.2], *Bessel functions of the second kind* are solutions of Bessel equations defined according to their asymptotics at infinity. To remove the restriction $\lambda \in \mathbb{L}^{n-1}$ on the definition of $J(z; \lambda; \xi)$, with ξ a $2n$ -th root of unity, we simply impose the additional condition

$$(7.4) \quad J(z; \lambda - \lambda e^n; \xi) = z^{n\lambda} J(z; \lambda; \xi).$$

REMARK 7.2. We may also use the following formula as an alternative definition of $J(z; \lambda; \xi)$ (compare [Qi1, Corollary 8.5])

$$(7.5) \quad J(z; \lambda; \xi) = \sqrt{n} \left(\frac{\pi}{2} \right)^{\frac{n-1}{2}} (-i\xi)^{\frac{n-1}{2}+|\lambda|} \sum_{l=1}^n (i\bar{\xi})^{n\lambda_l} S_l(\lambda) J_l(z; \varsigma, \lambda).$$

where ξ is an n -th root of $\varsigma 1$, conventionally, $(-i\xi)^{\frac{n-1}{2}+|\lambda|} = e^{(\frac{n-1}{2}+|\lambda|)(-\frac{1}{2}\pi i + i \arg \xi)}$ and $(i\bar{\xi})^{n\lambda_l} = e^{\frac{1}{2}\pi i n \lambda_l - i n \lambda_l \arg \xi}$, and $S_l(\lambda) = \prod_{k \neq l} \sin(\pi(\lambda_l - \lambda_k))^{-1}$.

Given an integer a , define $\xi_{a,j} = e^{2\pi i \frac{j+a-1}{n}}$, $j = 1, \dots, n$. Let $\sigma_{l,d}(\lambda)$, $d = 0, 1, \dots, n-1$, $l = 1, \dots, n$, denote the elementary symmetric polynomial in $e^{-2\pi i \lambda_1}, \dots, e^{-2\pi i \lambda_l}, \dots, e^{-2\pi i \lambda_n}$ of degree d . It follows from [Qi1, Corollary 8.7] that

$$(7.6) \quad J_l(z; +, \lambda) = \frac{e^{\frac{3}{4}\pi i((n-1)+2|\lambda|)}}{\sqrt{n}(2\pi)^{\frac{n-1}{2}}} e^{\pi i(\frac{1}{2}n+2a-2)\lambda_l} \sum_{j=1}^n (-)^{n-j} \xi_{a,j}^{-\frac{n-1}{2}-|\lambda|} \sigma_{l,n-j}(\lambda) J(z; \lambda; \xi_{a,j}).$$

In addition, we shall require the definition

$$\tau_l(\lambda) = \prod_{k \neq l} (e^{-2\pi i \lambda_m} - e^{-2\pi i \lambda_k}) = (-2i)^{n-1} e^{-\pi i |\lambda|} e^{-\pi i(n-2)\lambda_l} \prod_{k \neq l} \sin(\pi(\lambda_l - \lambda_k)).$$

We introduce the column vectors of the two kinds of Bessel functions

$$X(z; \lambda) = (J_l(z; +, \lambda))_{l=1}^n, \quad Y_a(z; \lambda) = (J(z; \lambda; \xi_{a,j}))_{j=1}^n,$$

and the matrices

$$\Sigma(\lambda) = (\sigma_{l,n-j}(\lambda))_{l,j=1}^n, \quad E_a(\lambda) = \text{diag} \left(e^{\pi i(\frac{1}{2}n+2a-2)\lambda_l} \right)_{l=1}^n, \quad D_a(\lambda) = \text{diag} \left((-)^{n-j} \xi_{a,j}^{-\frac{n-1}{2}-|\lambda|} \right)_{j=1}^n.$$

Then the formula (7.6) may be written as

$$(7.7) \quad X(z; \lambda) = \frac{e^{\frac{3}{4}\pi i((n-1)+2|\lambda|)}}{\sqrt{n}(2\pi)^{\frac{n-1}{2}}} \cdot E_a(\lambda) \Sigma(\lambda) D_a(\lambda) Y_a(z; \lambda).$$

We now formulate (7.3) as

$$(7.8) \quad J_{(\mu,m)}(z) = (-)^{|m|} e^{-\frac{1}{2}\pi i(n-1)} (4\pi^2)^{n-1} \cdot {}^t X \left(2\pi z^{\frac{1}{n}}; \lambda_{(\mu,m)}^+ \right) S_{(\mu,m)} X \left(2\pi \bar{z}^{\frac{1}{n}}; \lambda_{(\mu,m)}^- \right),$$

with $\lambda_{(\mu,m)}^\pm = \mu \pm \frac{1}{2}m$ and

$$S_{(\mu,m)} = \text{diag} \left(\tau_l \left(\lambda_{(\mu,m)}^\pm \right)^{-1} e^{-\pi i((n-2)\mu \mp m_l)} \right)_{l=1}^n.$$

We insert the formulae of $X \left(2\pi z^{\frac{1}{n}}; \lambda_{(\mu,m)}^+ \right)$ and $X \left(2\pi \bar{z}^{\frac{1}{n}}; \lambda_{(\mu,m)}^- \right)$ given by (7.7), with $\lambda = \lambda_{(\mu,m)}^+$, $a = 0$ in the former and $\lambda = \lambda_{(\mu,m)}^-$, $a = 1 - r$, for $r = 0, 1, \dots, n$, in the latter. Then

follows the formula

(7.9)

$$J_{(\mu, m)}(z) = (-)^{(n-1)+|m|} \frac{(2\pi)^{n-1}}{n} {}^t Y_0 \left(2\pi z^{\frac{1}{n}}; \lambda_{(\mu, m)}^+ \right) D_0 \left(\lambda_{(\mu, m)}^+ \right) {}^t \Sigma_{(\mu, m)} R_{(\mu, m)} \Sigma_{(\mu, m)} D_{1-r} \left(\lambda_{(\mu, m)}^- \right) Y_{1-r} \left(2\pi \bar{z}^{\frac{1}{n}}; \lambda_{(\mu, m)}^- \right),$$

where

$$\Sigma_{(\mu, m)} = \Sigma \left(\lambda_{(\mu, m)}^+ \right) = \Sigma \left(\lambda_{(\mu, m)}^- \right),$$

$$R_{(\mu, m)} = E_0 \left(\lambda_{(\mu, m)}^+ \right) S_{(\mu, m)} E_{1-r} \left(\lambda_{(\mu, m)}^- \right) = \text{diag} \left(\tau_l \left(\lambda_{(\mu, m)}^\pm \right)^{-1} e^{-2\pi i r \lambda_{(\mu, m), l}^\pm} \right)_{l=1}^n.$$

We are therefore reduced to computing the matrix ${}^t \Sigma_{(\mu, m)} R_{(\mu, m)} \Sigma_{(\mu, m)}$. For this, we have the following lemma.

LEMMA 7.3. *Let $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$ be a generic n -tuple in the sense that all its components are distinct. Let $\sigma_{l,d}$, respectively σ_d , denote the elementary symmetric polynomial in $x_1, \dots, \hat{x}_l, \dots, x_n$, respectively x_1, \dots, x_n , of degree d , and let $\tau_l = \prod_{h \neq l} (x_l - x_h)$. Define the matrices $\Sigma = (\sigma_{l, n-j})_{l,j=1}^n$, $X = \text{diag} (x_l)_{l=1}^n$ and $T = \text{diag} (\tau_l^{-1})_{l=1}^n$. Then, for any $r = 0, 1, \dots, n$, the matrix ${}^t \Sigma X^r T \Sigma$ can be written as*

$$\begin{pmatrix} (-)^{n-r} A & 0 \\ 0 & (-)^{n-r+1} B \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & \cdots & 0 & \sigma_n \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \sigma_{n-r+2} \\ \sigma_n & \cdots & \sigma_{n-r+2} & \sigma_{n-r+1} \end{pmatrix}, \quad B = \begin{pmatrix} \sigma_{n-r-1} & \sigma_{n-r-2} & \cdots & \sigma_0 \\ \sigma_{n-r-2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \sigma_0 & 0 & \cdots & 0 \end{pmatrix}.$$

More precisely, the (k, j) -th entry $a_{k,j}$, $k, j = 1, \dots, r$, of A is given by

$$a_{k,j} = \begin{cases} \sigma_{n+r-k-j+1} & \text{if } k+j \geq r+1, \\ 0 & \text{if otherwise,} \end{cases}$$

whereas the (k, j) -th entry $b_{k,j}$, $k, j = 1, \dots, n-r$, of B is given by

$$b_{k,j} = \begin{cases} \sigma_{n-r-k-j+1} & \text{if } k+j \leq n-r+1, \\ 0 & \text{if otherwise.} \end{cases}$$

PROOF OF LEMMA 7.3. Appealing to the Lagrange interpolation formula, we find in [Qil, Lemma 8.6] that the inverse of $T\Sigma$ is equal to the matrix $U = \left((-)^{n-j} x_l^{j-1} \right)_{j,l=1}^n$.

Therefore, it suffices to show that

$${}^t \Sigma X^r = \begin{pmatrix} (-)^{n-r} A & 0 \\ 0 & (-)^{n-r+1} B \end{pmatrix} U.$$

This is equivalent to the following two collections of identities,

$$\begin{aligned} \sum_{j=r-k+1}^r (-)^{r+j} \sigma_{n+r-k-j+1} x_l^{j-1} &= \sigma_{l,n-k} x_l^r, \quad k = 1, \dots, r, \\ \sum_{j=1}^{n-r-k+1} (-)^{j-1} \sigma_{n-r-k-j+1} x_l^{r+j-1} &= \sigma_{l,n-r-k} x_l^r, \quad k = 1, \dots, n-r, \end{aligned}$$

which are further equivalent to

$$\begin{aligned} \sum_{j=1}^k (-)^{k+j} \sigma_{n-j+1} x_l^{j-k-1} &= \sigma_{l,n-k}, \quad k = 1, \dots, r, \\ \sum_{j=1}^k (-)^{j-1} \sigma_{k-j} x_l^{j-1} &= \sigma_{l,k-1}, \quad k = 1, \dots, n-r. \end{aligned}$$

The last two identities can be easily seen, actually for all $k = 1, \dots, n$, from computing the coefficients of x^{k-1} and x^{2n-k} on the two sides of

$$\begin{aligned} \prod_{h \neq l} (x - x_h) &= \left(\sum_{p=0}^{\infty} x_l^p x^{-p-1} \right) \prod_{h=1}^n (x - x_h), \\ (x^n - x_l^n) \prod_{h \neq l} (x - x_h) &= \left(\sum_{p=1}^n x_l^{p-1} x^{n-p} \right) \prod_{h=1}^n (x - x_h), \end{aligned}$$

respectively.

Q.E.D.

Applying Lemma 7.3 with $x_l = e^{-2\pi i \lambda_{(\mu, \mathbf{m})}^{\pm}} = (-)^{m_l} e^{-2\pi i \mu_l}$ to the formula (7.9), we arrive at the following theorem.

THEOREM 7.4. *Let $(\mu, \mathbf{m}) \in \mathbb{L}^{n-1} \times \mathbb{Z}^n$ and $r \in \{0, 1, \dots, n\}$. Define $\xi_j = e^{2\pi i \frac{j-1}{n}}$, $\zeta_j = e^{2\pi i \frac{j-r}{n}}$, and denote by $\sigma_{(\mu, \mathbf{m})}^d$ the elementary symmetric polynomial in $(-)^{m_1} e^{-2\pi i \mu_1}$, ..., $(-)^{m_n} e^{-2\pi i \mu_n}$ of degree d , with $j = 1, \dots, n$ and $d = 0, 1, \dots, n$. Then we have*

$$\begin{aligned} J_{(\mu, \mathbf{m})}(z) &= (-)^{|\mathbf{m}|} \frac{(2\pi)^{n-1}}{n} \sum_{\substack{k,j=1,\dots,r \\ k+j \geq r+1}} C_{k,j}(\mu, \mathbf{m}) \\ &\quad J(2\pi z^{\frac{1}{n}}; \mu + \frac{1}{2}\mathbf{m}; \xi_k) J(2\pi \bar{z}^{\frac{1}{n}}; \mu - \frac{1}{2}\mathbf{m}; \zeta_j) \\ (7.10) \quad &+ (-)^{|\mathbf{m}|} \frac{(2\pi)^{n-1}}{n} \sum_{\substack{k,j=1,\dots,n-r \\ k+j \leq n-r+1}} D_{k,j}(\mu, \mathbf{m}) \\ &\quad J(2\pi z^{\frac{1}{n}}; \mu - \frac{1}{2}\mathbf{m}; \xi_{r+k}) J(2\pi \bar{z}^{\frac{1}{n}}; \mu - \frac{1}{2}\mathbf{m}; \zeta_{r+j}). \end{aligned}$$

with

$$(7.11) \quad C_{k,j}(\mu, \mathbf{m}) = (-)^{r+k+j+1} \xi_k^{-\frac{n-1}{2} - \frac{1}{2}|\mathbf{m}|} \zeta_j^{-\frac{n-1}{2} + \frac{1}{2}|\mathbf{m}|} \sigma_{(\mu, \mathbf{m})}^{n+r-k-j+1},$$

$$(7.12) \quad D_{k,j}(\mu, \mathbf{m}) = (-)^{r+k+j} \xi_{r+k}^{-\frac{n-1}{2} - \frac{1}{2}|\mathbf{m}|} \zeta_{r+j}^{-\frac{n-1}{2} + \frac{1}{2}|\mathbf{m}|} \sigma_{(\mu, \mathbf{m})}^{n-r-k-j+1}.$$

LEMMA 7.5. We retain the notations in Theorem 7.4. Moreover, we define $\Im(\boldsymbol{\mu}) = \max \{|\Im \mu_l|\}$.

(1.1). For $k = 1, \dots, r$, we have $C_{k,r-k+1}(\boldsymbol{\mu}, \mathbf{m}) = (-\bar{\xi}_k)^{|\mathbf{m}|}$.

(1.2). Let $k, j = 1, \dots, r$ be such that $k + j \geq r + 2$. Denote $p = k + j - r - 1$. We have the estimate

$$C_{k,j}(\boldsymbol{\mu}, \mathbf{m}) \leq \binom{n}{p} \exp(2\pi \min\{n - p, p\} \Im(\boldsymbol{\mu})).$$

(2.1). For $k = 1, \dots, n - r$, we have $D_{k,n-r-k+1}(\boldsymbol{\mu}, \mathbf{m}) = (-\bar{\xi}_{k+r})^{|\mathbf{m}|}$.

(2.2). Let $k, j = 1, \dots, n - r$ be such that $k + j \leq n - r$. Denote $p = n - r - k - j + 1$.

We have the estimate

$$D_{k,j}(\boldsymbol{\mu}, \mathbf{m}) \leq \binom{n}{p} \exp(2\pi \min\{n - p, p\} \Im(\boldsymbol{\mu})).$$

7.3. The rank-two case.

EXAMPLE 7.6. Let $\mu \in \mathbb{C}$ and $m \in \mathbb{Z}$.

If we define

$$(7.13) \quad J_{\mu,m}(z) = J_{-2\mu-\frac{1}{2}m}(z) J_{-2\mu+\frac{1}{2}m}(\bar{z}),$$

then

$$(7.14) \quad J_{(\mu,-\mu,m,0)}(z) = \begin{cases} \frac{2\pi^2}{\sin(2\pi\mu)} [\sqrt{z}]^{-m} (J_{\mu,m}(4\pi\sqrt{z}) - J_{-\mu,-m}(4\pi\sqrt{z})) & \text{if } m \text{ is even,} \\ \frac{2\pi^2 i}{\cos(2\pi\mu)} [\sqrt{z}]^{-m} (J_{\mu,m}(4\pi\sqrt{z}) + J_{-\mu,-m}(4\pi\sqrt{z})) & \text{if } m \text{ is odd,} \end{cases}$$

which should be interpreted in the way as in Theorem 7.1. We remark that the generic case is when $4\mu \notin 2\mathbb{Z} + m$.

On the other hand, using the connection formulae ([Wat, 3.61 (1, 2)])

$$J_\nu(z) = \frac{H_\nu^{(1)}(z) + H_\nu^{(2)}(z)}{2}, \quad J_{-\nu}(z) = \frac{e^{\pi i \nu} H_\nu^{(1)}(z) + e^{-\pi i \nu} H_\nu^{(2)}(z)}{2},$$

one obtains

$$(7.15) \quad J_{(\mu,-\mu,m,0)}(z) = \pi^2 i [\sqrt{z}]^{-m} \left(e^{2\pi i \mu} H_{\mu,m}^{(1)}(4\pi\sqrt{z}) + (-)^{m+1} e^{-2\pi i \mu} H_{\mu,m}^{(2)}(4\pi\sqrt{z}) \right),$$

with the definition

$$(7.16) \quad H_{\mu,m}^{(1,2)}(z) = H_{2\mu+\frac{1}{2}m}^{(1,2)}(z) H_{2\mu-\frac{1}{2}m}^{(1,2)}(\bar{z}).$$

8. The asymptotic expansion of $J_{(\mu,m)}(z)$

The asymptotic of $J_{(\mu,\delta)}(x)$ has already been established in [Qi1, Theorem 5.13, 9.4]. In the following, we shall present the asymptotic expansion of $J_{(\mu,m)}(z)$.

First of all, we have the following proposition on the asymptotic expansion of $J(z; \lambda; \xi)$, which is in substance [Qi1, Theorem 7.27].

PROPOSITION 8.1. Let $\lambda \in \mathbb{C}^n$ and define $\mathfrak{C} = \max \{|\lambda_l - \frac{1}{n}|\lambda|| + 1\}$. Let ξ be a $2n$ -th root of unity. For a small positive constant ϑ , say $0 < \vartheta < \frac{1}{2}\pi$, we define the sector

$$(8.1) \quad \mathbb{S}'_{\xi}(\vartheta) = \left\{ z : \left| \arg z - \arg(i\bar{\xi}) \right| < \pi + \frac{\pi}{n} - \vartheta \right\}.$$

For a positive integer A , we have the asymptotic expansion

$$(8.2) \quad J(z; \lambda; \xi) = e^{in\xi z} z^{-\frac{n-1}{2} - |\lambda|} \left(\sum_{\alpha=0}^{A-1} (i\xi)^{-\alpha} B_{\alpha} \left(\lambda - \frac{1}{n}|\lambda|e^n \right) z^{-\alpha} + O_{A,\vartheta,n}(\mathfrak{C}^{2A}|z|^{-A}) \right)$$

for all $z \in \mathbb{S}'_{\xi}(\vartheta)$ such that $|z| \gg_{M,\vartheta,n} \mathfrak{C}^2$. Here $B_{\alpha}(\lambda)$ is a certain symmetric polynomial function in $\lambda \in \mathbb{L}^{n-1}$ of degree 2α , with $B_0(\lambda) = 1$.

LEMMA 8.2. Let r be a positive integer. Suppose that either $n = 2r$ or $n = 2r - 1$. For a given constant $0 < \vartheta < \frac{1}{2n}\pi$ define the sector

$$(8.3) \quad \mathbb{S}_n(\vartheta) = \begin{cases} \left\{ z : -\frac{\pi}{2} - \frac{\pi}{n} + \vartheta < \arg z < -\frac{\pi}{2} + \frac{3\pi}{n} - \vartheta \right\} & \text{if } n = 2r, \\ \left\{ z : -\frac{\pi}{2} - \frac{\pi}{n} + \vartheta < \arg z < -\frac{\pi}{2} + \frac{2\pi}{n} - \vartheta \right\} & \text{if } n = 2r - 1, \end{cases}$$

Let $(\mu, \mathbf{m}) \in \mathbb{L}^{n-1} \times \mathbb{Z}^n$ and define $\mathfrak{C}(\mu, \mathbf{m}) = \max \{|\mu_l| + 1, |m_l - \frac{1}{n}|\mathbf{m}|| + 1\}$. Define $\xi_j = e^{2\pi i \frac{j-1}{n}}$ and $\zeta_j = e^{2\pi i \frac{j-r}{n}}$ for $j = 1, \dots, n$. Then, for any $z \in \mathbb{S}_n(\vartheta)$ such that $|z| \gg_{A,\vartheta,n} \mathfrak{C}(\mu, \mathbf{m})^2$, we have

$$J(2\pi z; \mu + \tfrac{1}{2}\mathbf{m}; \xi_k) J(2\pi \bar{z}; \mu - \tfrac{1}{2}\mathbf{m}; \zeta_j) = \frac{e(n(\xi_k z + \zeta_j \bar{z}))}{(2\pi)^{n-1}|z|^{n-1}[z]^{|\mathbf{m}|}} \left(\sum_{\substack{\alpha, \beta=0, \dots, A-1 \\ \alpha+\beta \leq A-1}} (i\xi_k)^{-\alpha} (i\zeta_j)^{-\beta} B_{\alpha, \beta}(\mu, \mathbf{m}) z^{-\alpha} \bar{z}^{-\beta} + O_{A,\vartheta,n}(\mathfrak{C}(\mu, \mathbf{m})^{2A}|z|^{-A}) \right),$$

with

$$(8.4) \quad B_{\alpha, \beta}(\mu, \mathbf{m}) = B_{\alpha} \left(\mu + \tfrac{1}{2}\mathbf{m} - \tfrac{1}{2n}|\mathbf{m}|e^n \right) B_{\beta} \left(\mu - \tfrac{1}{2}\mathbf{m} + \tfrac{1}{2n}|\mathbf{m}|e^n \right), \quad \alpha, \beta \in \mathbb{N},$$

where $B_{\alpha}(\lambda)$ is the polynomial function in λ of degree 2α given in Proposition 8.1.

PROOF. Recall that, for an integer a , we defined $\xi_{a,j} = e^{2\pi i \frac{j+a-1}{n}}$. Note that $\xi_j = \xi_{a,j}$ and $\zeta_j = \xi_{1-r,j}$. It is clear that

$$\bigcap_{j=1}^n \mathbb{S}'_{\xi_{a,j}}(\vartheta) = \left\{ z : -\frac{\pi}{2} - \frac{2a+1}{n}\pi + \vartheta < \arg z < -\frac{\pi}{2} - \frac{2a-3}{n}\pi + \vartheta \right\}.$$

We denote this sector by $\mathbb{S}'_a(\vartheta)$. Observe that, when $n = 2r$ or $2r - 1$, the intersection $\mathbb{S}'_0(\vartheta) \cap \overline{\mathbb{S}'_{1-r}(\vartheta)}$ is exactly the sector $\mathbb{S}_n(\vartheta)$. In other words, for all $j = 1, \dots, n$, $z \in \mathbb{S}'_{\xi_j}(\vartheta)$ and $\bar{z} \in \mathbb{S}'_{\zeta_j}(\vartheta)$ both hold if $z \in \mathbb{S}_n(\vartheta)$. Therefore, Proposition 8.1 can be applied to yield the asymptotic expansion of $J(2\pi z; \mu + \frac{1}{2}\mathbf{m}; \xi_k) J(2\pi \bar{z}; \mu - \frac{1}{2}\mathbf{m}; \zeta_j)$ as above. Q.E.D.

REMARK 8.3. Since the sector $\mathbb{S}_n(\vartheta)$ is of angle at least $\frac{3}{n}\pi - 2\vartheta > \frac{2}{n}\pi$, the sector $\mathbb{S}_n(\vartheta)^n = \{z^n : z \in \mathbb{S}_n(\vartheta)\}$ covers \mathbb{C}^\times .

In the case when n is odd, one may also choose $n = 2r + 1$. The sector $\mathbb{S}_n(\vartheta)$ should then be replaced by $e^{\frac{1}{n}\pi i} \cdot \mathbb{S}_n(\vartheta)$.

LEMMA 8.4. Let r be a positive integer and $n = 2r$ or $2r - 1$. Define $\xi_j = e^{2\pi i \frac{j-1}{n}}$, $\zeta_j = e^{2\pi i \frac{j-r}{n}}$, $j = 1, \dots, n$. For $0 < \vartheta < \frac{1}{2n}\pi$, let the sector $\mathbb{S}_n(\vartheta)$ be defined by (8.3) in Lemma 8.2.

(1.1). For $k = 1, \dots, r$, we have

$$\Im(\xi_k z + \zeta_{r-k+1} \bar{z}) = 0.$$

(1.2). Let $k, j = 1, \dots, r$ be such that $k + j \geq r + 2$. For any $z \in \mathbb{S}_n(\vartheta)$, we have

$$\Im(\xi_k z + \zeta_j \bar{z}) \geq 2 \sin\left(\frac{k+j-r-1}{n}\pi\right) \sin \vartheta \cdot |z|.$$

(2.1). For $k = 1, \dots, n-r$, we have

$$\Im(\xi_{k+r} z + \zeta_{n-k+1} \bar{z}) = 0.$$

(2.2). Let $k, j = 1, \dots, n-r$ be such that $k + j \leq n-r$. For any $z \in \mathbb{S}_n(\vartheta)$, we have

$$\Im(\xi_{k+r} z + \zeta_{j+r} \bar{z}) \geq \begin{cases} 2 \sin\left(\frac{n-r-k-j+1}{n}\pi\right) \sin \vartheta \cdot |z|, & \text{if } n = 2r, \\ 2 \sin\left(\frac{n-r-k-j+1}{n}\pi\right) \sin\left(\frac{\pi}{n} + \vartheta\right) \cdot |z|, & \text{if } n = 2r - 1. \end{cases}$$

PROOF. We shall only prove (1.1) and (1.2) in the case $n = 2r$. The other cases follow in exactly the same way.

Write $z = x e^{i\phi}$. Observe that the condition $z \in \mathbb{S}_{2r}(\vartheta)$ amounts to

$$(8.5) \quad \left| \phi + \frac{\pi}{2} - \frac{\pi}{2r} \right| < \frac{\pi}{r} - \vartheta.$$

Since

$$\begin{aligned} \xi_k z + \zeta_j \bar{z} &= x e^{2\pi i \frac{k-1}{2r} + i\phi} + x e^{2\pi i \frac{j-r}{2r} - i\phi} \\ &= x e^{\pi i \frac{k+j-r-1}{2r}} \left(e^{\pi i \frac{k-j+r-1}{2r} + i\phi} + e^{-\pi i \frac{k-j+r-1}{2r} - i\phi} \right), \end{aligned}$$

(1.1) is then obvious (we also note that $\zeta_{r-k+1} = \bar{\xi}_k$), whereas (1.2) is equivalent to

$$(8.6) \quad \cos\left(\frac{k-j+r-1}{2r}\pi + \phi\right) \geq \sin \vartheta.$$

Under our assumptions on k and j in (1.2), one has $|k-j| \leq r-2$. Along with (8.5), this yields the following estimate

$$\left| \frac{k-j+r-1}{2r}\pi + \phi \right| \leq \frac{r-2}{2r}\pi + \frac{\pi}{r} - \vartheta = \frac{\pi}{2} - \vartheta.$$

Thus (8.6) is proven.

Q.E.D.

REMARK 8.5. In cases other than those listed in Lemma 8.4, $\Im(\xi_k z + \xi_j \bar{z})$ can not always be nonnegative for all $z \in \mathbb{S}_n(\vartheta)$, regardless of the choice of $\vartheta \in (0, \frac{1}{2n}\pi)$. Fortunately, these cases are excluded from the second connection formula for $J_{(\mu, \mathbf{m})}(z)$ in Theorem 7.4.

Now the asymptotic expansion of $J_{(\mu, \mathbf{m})}(z)$ can be readily established using Theorem 7.4 along with Lemma 7.5, 8.2 and 8.4.

THEOREM 8.6. Denote by \mathbb{X}_n the set of n -th roots of unity. Let $(\mu, \mathbf{m}) \in \mathbb{L}^{n-1} \times \mathbb{Z}^n$ and define $\mathfrak{C}(\mu, \mathbf{m}) = \max \{|\mu_l| + 1, |m_l - \frac{1}{n}|\mathbf{m}|| + 1\}$. Let A be a positive integer. Then

$$(8.7) \quad J_{(\mu, \mathbf{m})}(z^n) = \sum_{\xi \in \mathbb{X}_n} \frac{e(n(\xi z + \bar{\xi} \bar{z}))}{n|z|^{n-1}[\xi z]^{|\mathbf{m}|}} \left(\sum_{\substack{\alpha, \beta=0, \dots, A-1 \\ \alpha+\beta \leq A-1}} i^{-\alpha-\beta} \xi^{-\alpha+\beta} B_{\alpha, \beta}(\mu, \mathbf{m}) z^{-\alpha} \bar{z}^{-\beta} \right) + O_{A,n}(\mathfrak{C}(\mu, \mathbf{m})^{2A} |z|^{-A-n+1}),$$

if $|z| \geq_{A,n} \mathfrak{C}(\mu, \mathbf{m})^2$, with the coefficient $B_{\alpha, \beta}(\mu, \mathbf{m})$ given by (8.4) in Lemma 8.2. Obviously, the main term in (8.7) is independent on the choice of the argument of z modulo $\frac{2}{n}\pi$.

9. Hankel transforms from the representation theoretic viewpoint

We shall start with a brief review of Hankel transforms over an *archimedean* local field^X in the work of Ichino and Templier [IT] on the Voronoï summation formula. For the theory of L -functions and local functional equations over a local field the reader is referred to Cogdell's survey [Cog]. We shall then discuss Hankel transforms using the Langlands classification. For this, Knapp's article [Kna] is used as our reference, with some change of notations for our convenience.

Let \mathbb{F} be an archimedean local field with normalized absolute value $\|\cdot\| = \|\cdot\|_{\mathbb{F}}$ defined as in §2.3, and let ψ be a given additive character on \mathbb{F} . If $s \in \mathbb{C}$, let ω_s denote the character $\omega_s(x) = \|x\|^s$. Let $\eta(x) = \text{sgn}(x)$ for $x \in \mathbb{R}^\times$ and $\eta(z) = [z]$ for $z \in \mathbb{C}^\times$.

Suppose for the moment $n \geq 2$. Let π be an infinite dimensional irreducible admissible generic representation of $\text{GL}_n(\mathbb{F})^{\text{XI}}$, and $\mathcal{W}(\pi, \psi)$ be the ψ -Whittaker model of π . Denote by ω_π the central character of π . Recall that the γ -factor $\gamma(s, \pi, \psi)$ of π is given by

$$\gamma(s, \pi, \psi) = \epsilon(s, \pi, \psi) \frac{L(1-s, \tilde{\pi})}{L(s, \pi)}$$

where $\tilde{\pi}$ is the contragradient representation of π , $\epsilon(s, \pi, \psi)$ and $L(s, \pi)$ are the ϵ -factor and the L -function of π respectively.

^XFor a nonarchimedean local field, Hankel transforms can also be constructed in the same way.

^{XI}Since π is a local component of an irreducible cuspidal automorphic representation in [IT], [IT] also assumes that π is unitary. However, if one only considers the local theory, this assumption is not necessary.

To a smooth compactly supported function w on \mathbb{F}^\times we associate a dual function \tilde{w} on \mathbb{F}^\times defined by [IT, (1.1)],

$$(9.1) \quad \begin{aligned} \int_{\mathbb{F}^\times} \tilde{w}(x) \chi(x)^{-1} \|x\|^{s-\frac{n-1}{2}} d^\times x \\ = \chi(-1)^{n-1} \gamma(1-s, \pi \otimes \chi, \psi) \int_{\mathbb{F}^\times} w(x) \chi(x) \|x\|^{1-s-\frac{n-1}{2}} d^\times x, \end{aligned}$$

for all s of real part sufficiently large and all unitary multiplicative characters χ of \mathbb{F}^\times . (9.1) is independent of the chosen Haar measure $d^\times x$ on \mathbb{F}^\times , and uniquely defines \tilde{w} in terms of π , ψ and w . We shall let the Haar measure be given as in §2.3. We call \tilde{w} the Hankel transform of w associated with π .

According to [IT, Lemma 5.1], there exists a smooth Whittaker function $W \in \mathcal{W}(\pi, \psi)$ so that

$$(9.2) \quad w(x) = W \begin{pmatrix} x & \\ & I_{n-1} \end{pmatrix},$$

for all $x \in \mathbb{F}^\times$. Denote by w_n the n -by- n permutation matrix whose anti-diagonal entries are 1, that is, the longest Weyl element of rank n , and define

$$w_{n,1} = \begin{pmatrix} 1 & & \\ & & \\ & & w_{n-1} \end{pmatrix}.$$

In the theory of integral representations of Rankin-Selberg L -functions, (9.1) amounts to the local functional equations of zeta integrals for $\pi \otimes \chi$, with

$$(9.3) \quad \tilde{w}(x) = \tilde{W} \begin{pmatrix} x & \\ & 1 \end{pmatrix} = W \left(w_2 \begin{pmatrix} x^{-1} & \\ & 1 \end{pmatrix} \right),$$

if $n = 2$, and

$$(9.4) \quad \tilde{w}(x) = \int_{\mathbb{F}^{n-2}} \tilde{W} \left(\begin{pmatrix} x & & \\ y & I_{n-2} & \\ & & 1 \end{pmatrix} w_{n,1} \right) dy_\psi,$$

if $n \geq 3$, where $\tilde{W} \in \mathcal{W}(\tilde{\pi}, \psi^{-1})$ is the dual Whittaker function defined by $\tilde{W}(g) = W(w_n \cdot {}^t g^{-1})$, for $g \in \mathrm{GL}_n(\mathbb{F})$, and dx_ψ denotes the self-dual additive Haar measure on \mathbb{F} with respect to ψ . See [IT, Lemma 2.3].

The constraint that π be *infinite dimensional* and *generic* is actually dispensable for defining the Hankel transform via (9.1). In the following, we shall assume that π is any irreducible admissible representation of $\mathrm{GL}_n(\mathbb{F})$. Moreover, we shall also include the case $n = 1$. It will be seen that, after renormalizing the functions w and \tilde{w} , the Hankel transform defined by (9.1) turns into the Hankel transform given by (4.34) or (4.43). For this, we shall apply the Langlands classification for irreducible admissible representation of $\mathrm{GL}_n(\mathbb{F})$.

9.1. Hankel transforms over \mathbb{R} . Suppose $\mathbb{F} = \mathbb{R}$. Recall that $\|\cdot\|_{\mathbb{R}} = |\cdot|$ is the ordinary absolute value. For $r \in \mathbb{R}^\times$ let $\psi(x) = \psi_r(x) = e(rx)$.

According to [Kna, §3, Lemma], every finite dimensional semisimple representation φ of the Weil group of \mathbb{R} may be decomposed into irreducible representations of dimension one or two. The one-dimensional representations are parametrized by $(\mu, \delta) \in \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$. We denote by $\varphi_{(\mu, \delta)}$ the representation given by (μ, δ) . $\varphi_{(\mu, \delta)}$ corresponds to the representation $\chi_{(\mu, \delta)} = \omega_\mu \eta^\delta$ of $\mathrm{GL}_1(\mathbb{R})$ under the Langlands correspondence over \mathbb{R} . The irreducible two-dimensional representations are parametrized by $(\mu, m) \in \mathbb{C} \times \mathbb{N}_+$. We denote by $\varphi_{(\mu, m)}$ the representation given by (μ, m) . $\varphi_{(\mu, m)}$ corresponds to the representation $\sigma(m) \otimes \omega_\mu(\det)$ of $\mathrm{GL}_2(\mathbb{R})$, where $\sigma(m)$ denotes the discrete series representation of weight m .

In view of the formulae [Kna, (3.6, 3.7)]^{XII} of L -functions and ϵ -factors, the definitions (2.3) and (2.6) of G_δ and G_m , along with the formula (2.10), one sees that

$$(9.5) \quad \gamma(s, \varphi_{(\mu, \delta)}, \psi) = \mathrm{sgn}(r)^\delta |r|^{s+\mu-\frac{1}{2}} G_\delta(1-s-\mu),$$

whereas

$$(9.6) \quad \gamma(s, \varphi_{(\mu, m)}, \psi) = \mathrm{sgn}(r)^{\delta(m)+1} |r|^{2s+2\mu-1} i G_m(1-s-\mu),$$

and

$$(9.7) \quad \begin{aligned} \gamma(s, \varphi_{(\mu, m)}, \psi) &= \gamma(s, \varphi_{(\mu+\frac{1}{2}m, \delta(m)+1)}, \psi) \gamma(s, \varphi_{(\mu-\frac{1}{2}m, 0)}, \psi) \\ &= \gamma(s, \varphi_{(\mu+\frac{1}{2}m, \delta(m))}, \psi) \gamma(s, \varphi_{(\mu-\frac{1}{2}m, 1)}, \psi). \end{aligned}$$

To $\varphi_{(\mu, m)}$ we shall attach either one of the following two parameters

$$(9.8) \quad (\mu + \tfrac{1}{2}m, \mu - \tfrac{1}{2}m, \delta(m) + 1, 0), \quad (\mu + \tfrac{1}{2}m, \mu - \tfrac{1}{2}m, \delta(m), 1).$$

REMARK 9.1. (9.7) reflects the isomorphism $\varphi_{(0, m)} \otimes \varphi_{(0, 1)} \cong \varphi_{(0, m)}$ of representations of the Weil group (here $(0, 1)$ is an element of $\mathbb{C} \times \mathbb{Z}/2\mathbb{Z}$), as well as the isomorphism $\sigma(m) \otimes \eta \cong \sigma(m)$ of representations of $\mathrm{GL}_2(\mathbb{R})$.

For φ reducible, $\gamma(s, \varphi, \psi)$ is the product of the γ -factors of irreducible constituents of φ . Suppose φ is n -dimensional. It follows from (9.5, 9.6, 9.7) that there is a parameter $(\mu, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$ attached to φ such that

$$(9.9) \quad \gamma(s, \varphi, \psi) = \mathrm{sgn}(r)^{|\delta|} |r|^{n(s-\frac{1}{2})+|\lambda|} G_{(\mu, \delta)}(1-s).$$

The irreducible constituents of φ is unique up to permutation, but, in view of the two different parameters attached to $\varphi_{(\mu, m)}$ in (9.8), the parameter (μ, δ) attached to φ may not.

Suppose that π corresponds to φ under the Langlands correspondence over \mathbb{R} . We have $\gamma(s, \pi, \psi) = \gamma(s, \varphi, \psi)$. It is known that π is an irreducible constituent of the principal representation unitarily induced from the character $\bigotimes_{l=1}^n \chi_{(\mu_l, \delta_l)}$ of the Borel subgroup. In particular, $\omega_\pi = \omega_{|\mu|} \eta^{|\delta|}$.

^{XII}The formulae in [Kna, (3.6, 3.7)] are for ψ_1 . The relation between the epsilon factors $\epsilon(s, \pi, \psi_r)$ and $\epsilon(s, \pi, \psi)$ is given in [Tat, §3] (see in particular [Tat, (3.6.6)]).

Now let $\chi = \chi_{(0,\delta)} = \eta^\delta$ in (9.1), $\delta \in \mathbb{Z}/2\mathbb{Z}$. In view of (9.9), one has the following expression of the γ -factor in (9.1),

$$(9.10) \quad \gamma(1-s, \pi \otimes \eta^\delta, \psi) = \omega_\pi(r) \left(\operatorname{sgn}(r)^\delta |r|^{\frac{1}{2}-s} \right)^n G_{(\mu, \delta + \delta e^n)}(s).$$

Some calculations show that (9.1) is exactly translated into (4.34) if one let

$$(9.11) \quad \begin{aligned} v(x) &= \omega_\pi(r) w(|r|^{-\frac{n}{2}} x) |x|^{-\frac{n-1}{2}}, \\ \Upsilon(x) &= \tilde{w}((-)^{n-1} \operatorname{sgn}(r)^n |r|^{-\frac{n}{2}} x) |x|^{-\frac{n-1}{2}}. \end{aligned}$$

Then, (4.39) can be reformulated as

$$(9.12) \quad \tilde{w}((-)^{n-1} x) = \omega_\pi(r) |r|^{\frac{n}{2}} |x|^{\frac{n-1}{2}} \int_{\mathbb{R}^\times} w(y) J_{(\mu, \delta)}(r^n xy) |y|^{1-\frac{n-1}{2}} d^\times y.$$

9.2. Hankel transforms over \mathbb{C} . Suppose $\mathbb{F} = \mathbb{C}$. Recall that $\|\cdot\|_{\mathbb{C}} = \|\cdot\| = |\cdot|^2$, where $|\cdot|$ denotes the ordinary absolute value. For $r \in \mathbb{C}^\times$ let $\psi(z) = \psi_r(z) = e(rz + \overline{r}z)$.

The Langlands classification and correspondence for $\mathrm{GL}_n(\mathbb{C})$ are less complicated. First of all, the Weil group of \mathbb{C} is simply \mathbb{C}^\times . Any n -dimensional semisimple representation φ of the Weil group \mathbb{C}^\times is the direct sum of one-dimensional representations. The one-dimensional representations are of the form $\chi_{(\mu, m)} = \omega_\mu \eta^m$, with $(\mu, m) \in \mathbb{C} \times \mathbb{Z}$. In view of the formulae [Kna, (4.6, 4.7)] of $L(s, \chi_{(\mu, m)})$ and $\epsilon(s, \chi_{(\mu, m)}, \psi)$ as well as the definition (2.6) of G_m , we have

$$(9.13) \quad \gamma(s, \chi_{(\mu, m)}, \psi) = [r]^m \|r\|^{s+\mu-\frac{1}{2}} G_m(1-s-\mu).$$

Thus φ is parametrized by some $(\mu, m) \in \mathbb{C}^n \times \mathbb{Z}^n$ and

$$(9.14) \quad \gamma(s, \varphi, \psi) = [r]^{|m|} \|r\|^{n(s-\frac{1}{2})+|\mu|} G_{(\mu, m)}(1-s).$$

This parametrization is unique up to permutation, in contrary to the case $\mathbb{F} = \mathbb{R}$.

If π corresponds to φ under the Langlands correspondence over \mathbb{C} , then one has $\gamma(s, \pi, \psi) = \gamma(s, \varphi, \psi)$. Moreover, π is an irreducible constituent of the principal representation unitarily induced from the character $\bigotimes_{l=1}^n \chi_{(\mu_l, m_l)}$ of the Borel subgroup. Therefore, $\omega_\pi = \omega_{|\mu|} \eta^{|m|}$.

Now let $\chi = \chi_{(0, m)} = \eta^m$ in (9.1), $m \in \mathbb{Z}$. Then (9.13) implies

$$(9.15) \quad \gamma(1-s, \pi \otimes \eta^m, \psi) = \omega_\pi(r) \left([r]^m \|r\|^{\frac{1}{2}-s} \right)^n G_{(\mu, m + me^n)}(s).$$

By putting

$$(9.16) \quad \begin{aligned} v(z) &= \omega_\pi(r) w(\|r\|^{-\frac{n}{2}} z) \|z\|^{-\frac{n-1}{2}}, \\ \Upsilon(z) &= \tilde{w}((-)^{n-1} [r]^{-n} \|r\|^{-\frac{n}{2}} z) \|z\|^{-\frac{n-1}{2}}, \end{aligned}$$

the identity (9.1) is translated into (4.43), and (4.49) can be reformulated as

$$(9.17) \quad \tilde{w}((-)^{n-1} z) = \omega_\pi(r) \|r\|^{\frac{n}{2}} \|z\|^{\frac{n-1}{2}} \int_{\mathbb{C}^\times} w(u) J_{(\mu, m)}(r^n zu) \|u\|^{1-\frac{n-1}{2}} d^\times u.$$

9.3. Some new notations. Let π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{F})$. For $\mathbb{F} = \mathbb{R}$, respectively $\mathbb{F} = \mathbb{C}$, if π is parametrized by (μ, δ) , respectively (μ, m) , we shall denote simply by J_π the Bessel kernel $J_{(\mu, \delta)}$, respectively $J_{(\mu, m)}$. Thus, (9.12) and (9.17) can be uniformly combined into one formula

$$(9.18) \quad \tilde{w}((-)^{n-1}x) = \omega_\pi(r) \|r\|^{\frac{n}{2}} \|x\|^{\frac{n-1}{2}} \int_{\mathbb{F}^\times} w(y) J_\pi(r^n xy) \|y\|^{1-\frac{n-1}{2}} d^\times y.$$

Proposition 4.17 (1) and 4.21 (1) are translated into the following lemma.

LEMMA 9.2. *Let π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{F})$, and let χ be a character on \mathbb{F}^\times . We have $J_{\chi \otimes \pi}(x) = \chi^{-1}(x) J_\pi(x)$.*

REMARK 9.3. *Let Z_n denote the center of GL_n . In view of Lemma 9.2, no generality will be lost if one only considers J_π for irreducible admissible representations π of $\mathrm{GL}_n(\mathbb{F})/Z_n(\mathbb{R}_+)$.*

Let φ be the n -dimensional semisimple representation of the Weil group of \mathbb{F} corresponding to π under the Langlands correspondence over \mathbb{F} .

If $\mathbb{F} = \mathbb{R}$, the function space $\mathcal{S}_{\mathrm{sis}}^{(\mu, \delta)}(\mathbb{R}^\times)$ depends on the choice of parameter (μ, δ) attached to φ , if some discrete series $\varphi_{(\mu, m)}$ occurs in its decomposition. Thus, one needs to redefine the function spaces for Hankel transforms according to the Langlands classification rather than the above parametrization. For this, let $n_1, n_2 \in \mathbb{N}$, $(\mu^1, \delta^1) \in \mathbb{C}^{n_1} \times (\mathbb{Z}/2\mathbb{Z})^{n_1}$ and $(\mu^2, m^2) \in \mathbb{C}^{n_2} \times \mathbb{N}_+^{n_2}$ be such that $n_1 + 2n_2 = n$ and $\varphi = \bigoplus_{l=1}^{n_1} \varphi_{(\mu_l^1, \delta_l^1)} \oplus \bigoplus_{l=1}^{n_2} \varphi_{(\mu_l^1, m_l^2)}$. We define the function space $\mathcal{S}_{\mathrm{sis}}^\pi(\mathbb{R}^\times) = \mathcal{S}_{\mathrm{sis}}^\varphi(\mathbb{R}^\times)$ to be

$$(9.19) \quad \mathcal{S}_{\mathrm{sis}}^{(-\mu^1, \delta^1)}(\mathbb{R}^\times) + \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \mathrm{sgn}(x)^\delta \mathcal{S}_{\mathrm{sis}}^{(-\mu^2 + \frac{1}{2}m^2, \mathbf{0})}(\mathbb{R}^\times),$$

where $\mathcal{S}_{\mathrm{sis}}^{(\mu, \delta)}(\mathbb{R}^\times)$ is defined by (4.32).

LEMMA 9.4. *$\mathcal{S}_{\mathrm{sis}}^\pi(\mathbb{R}^\times)$ is the sum of $\mathcal{S}_{\mathrm{sis}}^{(-\mu, \delta)}(\mathbb{R}^\times)$ for all the parameters (μ, δ) attached to π .*

PROOF. For $\delta \in \mathbb{Z}/2\mathbb{Z}$ and $j \in \mathbb{N}$, we have the inclusion

$$\mathrm{sgn}(x)^{\delta+\delta(m)} |x|^{\mu+\frac{1}{2}m} (\log |x|)^j \mathcal{S}(\mathbb{R}^\times) \subset \mathrm{sgn}(x)^\delta |x|^{\mu-\frac{1}{2}m} (\log |x|)^j \mathcal{S}(\mathbb{R}^\times).$$

It follows that

$$\begin{aligned} & \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \left(\mathrm{sgn}(x)^{\delta+\delta(m)} |x|^{\mu+\frac{1}{2}m} (\log |x|)^j \mathcal{S}(\mathbb{R}^\times) + \mathrm{sgn}(x)^{\delta+1} |x|^{\mu-\frac{1}{2}m} (\log |x|)^j \mathcal{S}(\mathbb{R}^\times) \right) \\ &= \sum_{\delta \in \mathbb{Z}/2\mathbb{Z}} \mathrm{sgn}(x)^\delta |x|^{\mu-\frac{1}{2}m} (\log |x|)^j \mathcal{S}(\mathbb{R}^\times). \end{aligned}$$

Then it is easy to verify this lemma by definitions. Q.E.D.

If $\mathbb{F} = \mathbb{C}$, we put

$$(9.20) \quad \mathcal{S}_{\mathrm{sis}}^\pi(\mathbb{C}^\times) = \mathcal{S}_{\mathrm{sis}}^\varphi(\mathbb{C}^\times) = \mathcal{S}_{\mathrm{sis}}^{(-\mu, -m)}(\mathbb{C}^\times).$$

Let $d = [\mathbb{F} : \mathbb{R}]$. For each character χ on $\mathbb{F}^\times / \mathbb{R}_+$ we define the Mellin transform \mathcal{M}_χ of a function $v \in \mathcal{S}_{\text{sis}}(\mathbb{F}^\times)$ by

$$(9.21) \quad \mathcal{M}_\chi v(s) = \int_{\mathbb{F}^\times} v(x) \chi(x) \|x\|^{\frac{1}{d}s} d^\times x.$$

THEOREM 9.5. *Let π be an irreducible admissible representation of $\text{GL}_n(\mathbb{F})$. Suppose $v \in \mathcal{S}_{\text{sis}}^\pi(\mathbb{F}^\times)$. Then there exists a unique $\tilde{v} \in \mathcal{S}_{\text{sis}}^{\tilde{\pi}}(\mathbb{F}^\times)$ satisfying the following identity*

$$\mathcal{M}_{\chi^{-1}} \tilde{v}(ds) = \gamma(1-s, \pi \otimes \chi, \psi_1) \mathcal{M}_\chi v(d(1-s))$$

for all characters χ on $\mathbb{F}^\times / \mathbb{R}_+$. We write $\mathcal{H}_\pi v = \tilde{v}$ and call \tilde{v} the normalized Hankel transform of v over \mathbb{F}^\times associated with π . Moreover, we have the Hankel inversion formula

$$\mathcal{H}_\pi v = \tilde{v}, \quad \mathcal{H}_{\tilde{\pi}} \tilde{v} = v.$$

PROOF. If $\mathbb{F} = \mathbb{R}$, this follows from Theorem 4.15, combined with Lemma 9.4. If $\mathbb{F} = \mathbb{C}$, this is simply a translation of Theorem 4.19. Q.E.D.

10. Bessel functions for $\text{GL}_2(\mathbb{F})$

Let $n = 2$ and retain the notations from §9 except for the different choice of the Weyl element $w_2 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, which is more often used for GL_2 in the literature.

Let π be an infinite dimensional irreducible admissible representation of $\text{GL}_2(\mathbb{F})^{\text{XIII}}$. Using (9.2, 9.3), one may rewrite (9.18) as follows,

$$(10.1) \quad W \begin{pmatrix} & 1 \\ -x^{-1} & \end{pmatrix} = \omega_\pi(r) \|r\| \int_{\mathbb{F}^\times} \|xy\|^{\frac{1}{2}} J_\pi(r^2 xy) W \begin{pmatrix} y & \\ & 1 \end{pmatrix} d^\times y,$$

for $W \in \mathcal{W}(\pi, \psi_r)$. We define

$$(10.2) \quad \mathcal{J}_{\pi, \psi_r}(x) = \omega_\pi(r) \|r\| \sqrt{\|x\|} J_\pi(r^2 x).$$

We call $\mathcal{J}_{\pi, \psi}(x)$ the *Bessel function associated with π and ψ* . (10.1) then reads

$$(10.3) \quad W \begin{pmatrix} & 1 \\ -x^{-1} & \end{pmatrix} = \int_{\mathbb{F}^\times} \mathcal{J}_{\pi, \psi}(xy) W \begin{pmatrix} y & \\ & 1 \end{pmatrix} d^\times y.$$

Moreover, with the observation

$$W \begin{pmatrix} & 1 \\ -x^{-1} & \end{pmatrix} = \omega_\pi(-x)^{-1} W \left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} w_2 \right),$$

(10.3) turns into

$$(10.4) \quad W \left(\begin{pmatrix} x & \\ & 1 \end{pmatrix} w_2 \right) = \omega_\pi(-x) \int_{\mathbb{F}^\times} \mathcal{J}_{\pi, \psi}(xy) W \begin{pmatrix} y & \\ & 1 \end{pmatrix} d^\times y. \text{XIV}$$

^{XIII}It is well-known that a representation of $\text{GL}_2(\mathbb{F})$ satisfying these conditions is generic.

^{XIV}In the real case, this identity is given in [CPS, Theorem 4.1].

Thus, (10.4) indicates that the action of the Weyl element w_2 on the Kirillov model

$$\mathcal{K}(\pi, \psi) = \left\{ w(x) = W \begin{pmatrix} x \\ 1 \end{pmatrix} : W \in \mathcal{W}(\pi, \psi) \right\}$$

is essentially a Hankel transform. From this perspective, the Hankel inversion formula follows from the simple identity $w_2^2 = I_2$. This may be seen from the following lemma.

LEMMA 10.1. *Let π be an irreducible admissible representation of $\mathrm{GL}_2(\mathbb{F})$. Then we have $J_{\tilde{\pi}}(x) = \omega_{\pi}(x)J_{\pi}(x)$.*

PROOF. This follows from some straightforward calculations using Proposition 4.17 (1) and 4.21 (1). Q.E.D.

REMARK 10.2. *The representation theoretic viewpoint of Lemma 10.1 is the isomorphism $\tilde{\pi} \cong \omega^{-1} \otimes \pi$. With this, Lemma 10.1 is a direct consequence of Lemma 9.2.*

Finally, we shall summarize the formulae for the Bessel functions associated with infinite dimensional irreducible *unitary* representations of $\mathrm{GL}_2(\mathbb{F})$. First of all, in view of Lemma 9.2 and Remark 9.3, one may assume without loss of generality that π is trivial on $Z_2(\mathbb{R}_+)$. Moreover, with the simple observation

$$(10.5) \quad \mathcal{J}_{\pi, \psi_r}(x) = \omega_{\pi}(r) \mathcal{J}_{\pi, \psi_1}(r^2 x),$$

it is sufficient to consider the Bessel function $\mathcal{J}_{\pi} = \mathcal{J}_{\pi, \psi_1}$ associated with ψ_1 .

10.1. Bessel functions for $\mathrm{GL}_2(\mathbb{R})$. Under the Langlands correspondence, we have the following classification of infinite dimensional irreducible unitary representations of $\mathrm{GL}_2(\mathbb{R})/Z_2(\mathbb{R}_+)$.

- (principal series and the limit of discrete series) $\varphi_{(it, \epsilon + \delta)} \oplus \varphi_{(-it, \epsilon)}$, with $t \in \mathbb{R}$ and $\epsilon, \delta \in \mathbb{Z}/2\mathbb{Z}$,
- (complementary series) $\varphi_{(t, \epsilon)} \oplus \varphi_{(-t, \epsilon)}$, with $t \in (0, \frac{1}{2})$ and $\epsilon \in \mathbb{Z}/2\mathbb{Z}$,
- (discrete series) $\varphi_{(0, m)}$, with $m \in \mathbb{N}_+$.

Here, in the first case, the corresponding representation is a limit of discrete series if $t = 0$ and $\delta = 1$ and a principal series representation if otherwise. We shall write the corresponding representations as $\eta^{\epsilon} \otimes \pi^+(it)$ if $\delta = 0$, $\eta^{\epsilon} \otimes \pi^-(it)$ if $\delta = 1$, $\eta^{\epsilon} \otimes \pi(t)$ and $\sigma(m)$, respectively. We have

$$(10.6) \quad \omega_{\pi^+(it)} = 1, \quad \omega_{\pi^-(it)} = \eta, \quad \omega_{\pi(t)} = 1, \quad \omega_{\sigma(m)} = \eta^{m+1}.$$

Furthermore, we have the equivalences $\pi^+(it) \cong \pi^+(-it)$ and $\pi^-(it) \cong \eta \otimes \pi^+(-it)$.

As a consequence of Example 4.18, we have the following proposition.

PROPOSITION 10.3.

(1). *Let $t \in \mathbb{R}$. We have for $x \in \mathbb{R}_+$*

$$\begin{aligned} \mathcal{J}_{\pi^+(it)}(x) &= \frac{\pi i}{\sinh(\pi t)} \sqrt{x} (J_{2it}(4\pi \sqrt{x}) - J_{-2it}(4\pi \sqrt{x})), \\ \mathcal{J}_{\pi^+(it)}(-x) &= 4 \cosh(\pi t) \sqrt{x} K_{2it}(4\pi \sqrt{x}), \end{aligned}$$

where it is understood that when $t = 0$ the right hand side of the first formula should be replaced by its limit, and

$$\begin{aligned}\mathcal{J}_{\pi^-(it)}(x) &= \frac{\pi i}{\cosh(\pi t)} \sqrt{x} (J_{2it}(4\pi \sqrt{x}) + J_{-2it}(4\pi \sqrt{x})), \\ \mathcal{J}_{\pi^-(it)}(-x) &= 4 \sinh(\pi t) \sqrt{x} K_{2it}(4\pi \sqrt{x}).\end{aligned}$$

(2). Let $t \in (0, \frac{1}{2})$. We have for $x \in \mathbb{R}_+$

$$\begin{aligned}\mathcal{J}_{\pi(t)}(x) &= -\frac{\pi}{\sin(\pi t)} \sqrt{x} (J_{2t}(4\pi \sqrt{x}) - J_{-2t}(4\pi \sqrt{x})), \\ \mathcal{J}_{\pi(t)}(-x) &= 4 \cos(\pi t) \sqrt{x} K_{2t}(4\pi \sqrt{x}).\end{aligned}$$

(3). Let $m \in \mathbb{N}_+$. We have for $x \in \mathbb{R}_+$

$$\mathcal{J}_{\sigma(m)}(x) = 2\pi i^{m+1} \sqrt{x} J_m(4\pi \sqrt{x}), \quad \mathcal{J}_{\sigma(m)}(-x) = 0.$$

REMARK 10.4. $\eta^\epsilon \otimes \pi^+(it)$, $\eta^\epsilon \otimes \pi(t)$ and $\sigma(2d-1)$ exhaust all the infinite dimensional irreducible unitary representations of $\mathrm{PGL}_2(\mathbb{R})$. Their Bessel functions are also given in [CPS, Proposition 6.1].

10.2. Bessel functions for $\mathrm{GL}_2(\mathbb{C})$. Under the Langlands correspondence, we have the following classification of infinite dimensional irreducible unitary representations of $\mathrm{GL}_2(\mathbb{C})/\mathrm{Z}_2(\mathbb{R}_+)$.

- (principal series) $\chi_{(it, k+d+\delta)} \oplus \chi_{(-it, k-d)}$, with $t \in \mathbb{R}$, $k, d \in \mathbb{Z}$ and $\delta \in \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$,
- (complementary series) $\chi_{(t, k+d)} \oplus \chi_{(-t, k-d)}$, with $t \in (0, \frac{1}{2})$, $k \in \mathbb{Z}$ and $d \in \mathbb{Z}$.

We write the corresponding representations as $\eta^k \otimes \pi_d^+(it)$ if $\delta = 0$, $\eta^k \otimes \pi_d^-(it)$ if $\delta = 1$ and $\eta^k \otimes \pi_d(t)$, respectively. We have

$$(10.7) \quad \omega_{\pi_d^+(it)} = 1, \quad \omega_{\pi_d^-(it)} = \eta, \quad \omega_{\pi_d(t)} = 1.$$

Furthermore, we have the equivalences $\pi_d^+(it) \cong \pi_{-d}^+(-it)$, $\pi_d^-(it) \cong \pi_{-d-1}^-(-it)$.

According to Example 7.6, we have the following proposition.

PROPOSITION 10.5. Recall the definitions (7.13, 7.16) of $J_{\mu, m}(z)$ and $H_{\mu, m}^{(1,2)}(z)$ in Example 7.6.

(1). Let $t \in \mathbb{R}$ and $d \in \mathbb{Z}$. We have for $z \in \mathbb{C}^\times$

$$\begin{aligned}\mathcal{J}_{\pi_d^+(it)}(z) &= -\frac{2\pi^2 i}{\sinh(2\pi t)} |z| (J_{it, 2d}(4\pi \sqrt{z}) - J_{-it, -2d}(4\pi \sqrt{z})) \\ &= \pi^2 i |z| \left(e^{-2\pi t} H_{it, 2d}^{(1)}(4\pi \sqrt{z}) - e^{2\pi t} H_{it, 2d}^{(2)}(4\pi \sqrt{z}) \right), \\ \mathcal{J}_{\pi_d^-(it)}(z) &= \frac{2\pi^2 i}{\cosh(2\pi t)} \sqrt{|z| \bar{z}} (J_{it, 2d+1}(4\pi \sqrt{z}) + J_{-it, -2d-1}(4\pi \sqrt{z})) \\ &= \pi^2 i \sqrt{|z| \bar{z}} \left(e^{-2\pi t} H_{it, 2d+1}^{(1)}(4\pi \sqrt{z}) + e^{2\pi t} H_{it, 2d+1}^{(2)}(4\pi \sqrt{z}) \right).\end{aligned}$$

(2). Let $t \in (0, \frac{1}{2})$ and $d \in \mathbb{Z}$. We have for $z \in \mathbb{C}^\times$

$$\begin{aligned} \mathcal{J}_{\pi_d(t)}(z) &= \frac{2\pi^2}{\sin(2\pi t)} |z| (J_{t,2d}(4\pi\sqrt{z}) - J_{-t,-2d}(4\pi\sqrt{z})) \\ &= \pi^2 i |z| \left(e^{2\pi i t} H_{t,2d}^{(1)}(4\pi\sqrt{z}) - e^{-2\pi i t} H_{t,2d}^{(2)}(4\pi\sqrt{z}) \right). \end{aligned}$$

In view of Corollary 6.16, we have the following integral representations of $\mathcal{J}_\pi(xe^{i\phi})$ except for $\pi = \pi_d(t)$ and $t \in [\frac{3}{8}, \frac{1}{2})$.

PROPOSITION 10.6.

(1). Let $t \in \mathbb{R}$ and $d \in \mathbb{Z}$. We have for $x \in \mathbb{R}_+$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$

$$\begin{aligned} \mathcal{J}_{\pi_d^+(it)}(xe^{i\phi}) &= 4\pi(-1)^d x e^{id\phi} \int_0^\infty y^{4it-1} [y^{-1} + ye^{i\phi}]^{-2d} J_{2d}(4\pi\sqrt{x}|y^{-1} + ye^{i\phi}|) dy, \\ \mathcal{J}_{\pi_d^-(it)}(xe^{i\phi}) &= 4\pi i(-1)^d x e^{id\phi} \int_0^\infty y^{4it-1} [y^{-1} + ye^{i\phi}]^{-2d-1} J_{2d+1}(4\pi\sqrt{x}|y^{-1} + ye^{i\phi}|) dy. \end{aligned}$$

(2). Let $t \in (0, \frac{3}{8})$ and $d \in \mathbb{Z}$. We have for $x \in \mathbb{R}_+$ and $\phi \in \mathbb{R}/2\pi\mathbb{Z}$

$$\mathcal{J}_{\pi_d(t)}(xe^{i\phi}) = 4\pi(-1)^d x e^{id\phi} \int_0^\infty y^{4t-1} [y^{-1} + ye^{i\phi}]^{-2d} J_{2d}(4\pi\sqrt{x}|y^{-1} + ye^{i\phi}|) dy.$$

The integral on the right hand side converges absolutely only for $t \in (0, \frac{1}{8})$.

REMARK 10.7. $\pi_d^+(it)$ and $\pi_d(t)$ exhaust all the infinite dimensional irreducible unitary representations of $\mathrm{PGL}_2(\mathbb{C})$. Proposition 10.5 shows that the Bessel function for $\pi_d^+(it)$ actually coincide with that given in [BM]. More precisely, we have the equality $\mathcal{J}_{\pi_d^+(it)}(z) = 2\pi^2 |z| \mathcal{K}_{2it,-d}(4\pi\sqrt{z})$, with $\mathcal{K}_{v,p}$ given by [BM, (6.21), (7.21)]. Furthermore, the integral representation of $\mathcal{J}_{\pi_d^+(it)}$ in Proposition 10.6 (1) is tantamount to [BM, Theorem 12.1]. We have similar relations between the Bessel function for $\pi_d^-(it)$ and that given in [LG].

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